Quantum Mechanics is *derivable* from Special Relativity. Using the 4-Vectors of SR, plus a few empirical facts, one can derive the principles of QM. Therefore, the standard QM "axioms" are actually emergent from the rules of SR. Hence: [**SR→QM**]
The 4D <Time·Space>-splitting into 1D Temporal + 3D Spatial components plays an integral role in understanding these relations.

KeyWords: Special Relativity (SR), Quantum Mechanics (QM), Quantum-Classical relation, Quantum Emergence, SRQM, SR→QM

Introduction:
There are currently two main foundational bodies of physics theory used to model reality as we know it:
Relativity {General (GR) ~ astrophysical scale + Special (SR) ~ connects all scales} & **Quantum Mechanics** (QM) ~ atomic scale

(1) Relativity uses tensor mathematics to calculate the properties of 4D Spacetime. GR is used for physical systems in which the mass of objects is very large. Essentially it gravitationally says: "Mass tells spacetime how to curve, spacetime tells mass how to move". SR is a "special" limiting-case of GR for low curvature ~ low mass. SR shows that certain physical properties once thought totally separate are actually dual to one another: (time:space), (energy:momentum), etc. Tensor mathematics has the concept that tensor component-values can relativistically vary such that measurements between events are independent of arbitrarily-imposed coordinate-systems. (ex. consider 2D pictures of a 3D object from different angles & distances: the pics look different, yet are all the same object) Integral to physical, relativistic tensors is that {4D <Time·Space> = 1D Temporal (t) + 3D Spatial (x,y,z)} entities have specific ways of splitting into their various natural, measurable components, depending on the type of tensor that they are represented by:

4-Scalar	$S = S$	(1) Invariant Lorentz Scalar, same for all frames {s} or $\{s_o\}$	1	{4D (0,0)-Tensor} component
4-Vector	$\mathbf{V} = V^\mu$	$(1\ ^0 + 3\ ^j)$-splitting into $\{v^t, v^x, v^y, v^z\}$	4	{4D (1,0)-Tensor} components
4-Tensor,$_{\text{AntiSymmetric}}$	$\boldsymbol{T}_{asym} = T_{asym}^{\mu\nu}$	$(3\ ^{0j} + 3\ ^{j\neq k})$-splitting into $\{t^{tx}, t^{ty}, t^{tz}, t^{xy}, t^{xz}, t^{yz}\}$ w/ all $^{j=k}$ comps=0	6	{4D (2,0)-Tensor} components
4-Tensor,$_{\text{Symmetric}}$	$\boldsymbol{T}_{sym} = T_{sym}^{\mu\nu}$	$(1\ ^{00} + 3\ ^{0j} + 3\ ^{j=k} + 3\ ^{j\neq k})$-splitting into $\{t^{tt}, t^{tx}, t^{ty}, t^{tz}, t^{xx}, t^{yy}, t^{zz}, t^{xy}, t^{xz}, t^{yz}\}$	10	{4D (2,0)-Tensor} components

There are relativistic Symmetries/Operations in nature which leave the interval-measurement between events unchanged (invariant) and lead to fundamental Conservation Laws. These can use active or passive transformations, including changes of coordinate basis. SR 4-Vectors have a Poincaré Group linear mapping ($V^{\mu'} = \Lambda^{\mu'}{}_\nu V^\nu + \Delta V[\Delta X^{\mu'}]$) which preserves interval-magnitude: ($V^{\mu'} V_{\mu'} = V^\nu V_\nu$). The Poincaré Group {Lorentz Group $\Lambda^\mu{}_\nu$ + SpaceTime Translation Group ΔX^μ} is the Full SpaceTime Symmetry Group and provides the (10) Isometries which match the {AntiSymmetric 4D (2,0)-Tensor [3+3]-splitting→(6) + 4D (1,0)-Tensor (1+3)-splitting→(4)}.

[↻]

Lorentz Group ($\Lambda^\mu{}_\nu$) Symmetry → Conservation of 4-AngularMomentum $\boldsymbol{M} = M^{\mu\nu} = R^\mu \wedge P^\nu = [[m^{\mu\nu}]] = [[0,-c\mathbf{n}],[c\mathbf{n}^T,\mathbf{l}=\mathbf{r}\wedge\mathbf{p}]]$:Isotropy
The spatial part: 3 Space-Space-Rotation ($\Lambda^\mu{}_\nu \to R^\mu{}_\nu$) Symmetry → Conservation of 3-angular-momentum $\mathbf{l} = l^k$ same all directions
The mixed part: 3 Space-Time-Boost ($\Lambda^\mu{}_\nu \to B^\mu{}_\nu$) Symmetry → Conservation of 3-mass-moment $\mathbf{n} = n^k$ θ,φ (∗)

[→]

Spacetime-Translation Group (ΔX^μ) Symmetry → Conservation of 4-LinearMomentum $\mathbf{P} = P^\mu = (p^\mu) = (p^0, p^k) = (E/c, \mathbf{p})$: Homogeneity
The temporal part: 1 Time-Translation ($\Delta x^0 = c\Delta t$) Symmetry → Conservation of energy $E = cp^0$ same all extent
The spatial part: 3 Space-Translation ($\Delta x^k = \Delta\mathbf{x}$) Symmetry → Conservation of 3-momentum $\mathbf{p} = p^k$ $\Delta\mathbf{X}$ [▦]

P: the 4-Momentum & **W**: the Pauli–Lubanski spin pseudovector. give $(\mathbf{P}\cdot\mathbf{P}) \to$ mass (m_o) and $(\mathbf{W}\cdot\mathbf{W}) \to$ spin (s_o), which are the two Casimir Invariants of the Poincaré Group, i.e. the quantities that commute with all generators of the Poincaré Group $\mathbf{R}^{1,3} \ltimes O(1,3)$.

(2) Quantum Mechanics typically uses an operator formalism, Hilbert space, and wavefunctions to describe the properties of fundamental particles and their interactions. Many curious and obscure properties arise: non-zero commutation of measurements, wave-particle duality, matter-waves, quantization of energy-levels, superposition of states, single-particle interference effects, unitary time-evolution, Born Probability, Bell's Theorem, Heisenberg Uncertainty, CPT Symmetry, entanglement, B-E:F-D Statistics, etc. QM has internal symmetry principles, with all quantum particles obeying some portion of a universal { U(1) × SU(2) × SU(3) } Symmetry.

For Relativity, Newtonian physics emerges when {GR limit-case low-curvature $g^{\mu\nu} \to \eta^{\mu\nu}$} and {SR limit-case low-velocity $|\mathbf{v}| \ll c$}.
For QM, Newtonian physics emerges when {QM limit-case low-divergence $\hbar|\nabla\cdot\mathbf{p}| \ll (\mathbf{p}\cdot\mathbf{p})$ or $|\nabla\cdot\mathbf{k}| \ll (\mathbf{k}\cdot\mathbf{k})$ or $S_{action} \gg \hbar$}.

For many decades, physicists have been trying to unite these two main theories. While QM appears to be totally compatible with SR in the form of Relativistic Quantum Mechanics (RQM), it does not "seem" to be compatible with GR, as their mathematical structures "seem" different. The main attempts at unification have been to assume QM as fundamental and try to "quantize gravity" in various ways, i.e. to impose the mathematical rules of QM onto that of GR, basically, to-date, without success.

In this work, a novel pathway, [**SRQM**] or [**SR→QM**], will be shown that unites the two theories in a way that is mathematically elegant and precise, and that explains why the previous attempts have been unsuccessful. Essentially, GR is taken as fundamental instead of QM. SRQM uses the mathematics of 4-Vectors and other 4D Tensors to describe the properties and relations of physical objects and concepts of both relativistic and quantum physics. Quantum "axioms" are instead actually emergent SR-derived principles. Consider the following idea, with each level a special limiting-case subset of the more comprehensive physical theory:
GR → {limit-case $g^{\mu\nu} \to \eta^{\mu\nu}$} → SR → derives → RQM → {limit-case $|\mathbf{v}| \ll c$} → QM → {limit-case $\hbar|\nabla\cdot\mathbf{p}| \ll (\mathbf{p}\cdot\mathbf{p})$} → (EM & CM)
The main idea of [**SRQM**] is that the rules of RQM and QM can be **derived** from the rules of SR. Thus: [**SR→QM**]

Notation / Conventions / Fundamentals:

Tensor Convention {Temporal, 0^{th} Component, Positive(+), SI} = Metric Signature (+,-,-,-) with [SI Dimensional-Units].
aka. {"Time-Positive", "Particle-Physics", "West-Coast", "Mostly-Minuses"} Metric Sign Convention → The "Metric System" :-)

SR <Time·Space>-splitting Component Coloring Mnemonic: Temporal (blue) + Spatial (red) give Mixed SpaceTime (purple)

4D "Flat" <Time·Space> SR:Minkowski Metric $\eta_{\mu\nu} = \eta^{\mu\nu} \rightarrow \text{Diagonal}[+1,-1,-1,-1]_{(\text{Cartesian})}$: Generally, $\{g_{\mu\nu}\}=1/\{g^{\mu\nu}\}$ for non-zero
Mixed 4D (1,1)-Tensor form Minkowski Metric $\eta^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} = \text{Diagonal}[+1,+1,+1,+1]_{(\text{Always})} = I_{[4]} = g^{\mu}{}_{\nu}$ = Kronecker Delta = Identity

4-Position $\mathbf{R} = R^{\mu} = (ct,\mathbf{r}) = \eta^{\mu\nu}R_{\nu}$: 4D Position-OneForm $R_{\mu} = (ct,-\mathbf{r}) = \eta_{\mu\nu}R^{\nu}$ [m]
4-Gradient $\boldsymbol{\partial} = \partial^{\mu} = (\partial_t/c,-\nabla) = (\partial/\partial R_{\mu}) = \eta^{\mu\nu}\partial_{\nu}$: 4D Gradient-OneForm $\partial_{\mu} = (\partial_t/c,\nabla) = (\partial/\partial R^{\mu}) = \eta_{\mu\nu}\partial^{\nu}$ [1/m]

$\partial^{\mu}[R^{\nu}] = \boldsymbol{\partial}[\mathbf{R}] = (\partial_t/c,-\nabla)[(ct,\mathbf{r})] \rightarrow (\partial_t/c,-\partial_x,-\partial_y,-\partial_z)[(ct,x,y,z)]$ = Jacobian = $\text{Diagonal}[+1,-1,-1,-1]_{(\text{Cartesian})} = \eta^{\mu\nu}$ = SR:Minkowski Metric
$\partial^{\mu}\,\eta_{\mu\nu}\,R^{\nu} = (\boldsymbol{\partial}\cdot\mathbf{R}) = (\partial_t/c,-\nabla)\cdot(ct,\mathbf{r}) \rightarrow (\partial_t/c,-\partial_x,-\partial_y,-\partial_z)\cdot(ct,x,y,z) = (\partial_t ct/c+\partial_x x+\partial_y y+\partial_z z) = 4$ = SR 4D SpaceTime Dimension
$\partial_{\nu}[R^{\mu'}] = \partial R^{\mu'}/\partial R^{\nu} = \Lambda^{\mu'}{}_{\nu}$ = SR Lorentz Transformation: $A^{\mu'} = \Lambda^{\mu'}{}_{\nu}A^{\nu}$ = New.Ref.Frame' = LorentzTransf. contracted w/ Old.Ref.Frame

4D Tensors use Greek indices: ex. { μ, ν, σ, ρ, ...} : ex. 4-Position $R^{\mu} = (r^{\mu}) = (r^0,r^1,r^2,r^3)$, with 4 possible index-values {0,1,2,3}
3D tensors use Latin indices: ex. { i, j, k, …} : ex. 3-position $r^k = (r^k) = (r^1,r^2,r^3)$, with 3 possible index-values {1,2,3}

4-Vector (4D) $\mathbf{A} = A^{\mu} = (a^{\mu}) = (a^0,\mathbf{a}) = (a^0,a^k) = (a^0,a^1,a^2,a^3) \rightarrow (a^t,a^x,a^y,a^z)_{[\text{Cartesian:rectangular}]} \rightarrow (a^t,a^r,a^{\theta},a^{\varphi})_{[\text{spherical}]}$ → other coordinate basis
3-vector (3D) $\mathbf{a} = a^k = (a^k) = (\mathbf{a}) = (a^k) = (a^1,a^2,a^3) \rightarrow (a^x,a^y,a^z)_{[\text{Cartesian:rectangular}]} \rightarrow (a^r,a^{\theta},a^{\varphi})_{[\text{spherical}]}$ → other coordinate basis

4-Scalar	S = S	(1) Invariant Lorentz Scalar, same for all frames, {s} or $\{s_o\}$	1 {4D (0,0)-Tensor} component
4-Vector	$\mathbf{V} = V^{\mu}$	($1^{\,0}+3^{\,j}$)-splitting into $\{v^t,v^x,v^y,v^z\}$	4 {4D (1,0)-Tensor} components
4-Tensor,$_{\text{AntiSymmetric}}$	$\boldsymbol{T_{asym}} = T_{asym}{}^{\mu\nu}$	($3^{\,0j}+3^{\,j\neq k}$)-splitting into $\{t^{tx},t^{ty},t^{tz},t^{xy},t^{xz},t^{yz}\}$ w/ all $^{j=k}$ comps=0	6 {4D (2,0)-Tensor} components
4-Tensor,$_{\text{Symmetric}}$	$\boldsymbol{T_{sym}} = T_{sym}{}^{\mu\nu}$	($1^{\,00}+3^{\,0j}+3^{\,j=k}+3^{\,j\neq k}$)-splitting into $\{t^{tt},t^{tx},t^{ty},t^{tz},t^{xx},t^{yy},t^{zz},t^{xy},t^{xz},t^{yz}\}$	10 {4D (2,0)-Tensor} components

S = {s} = number $(1=4^0)$: $\mathbf{V} = V^{\mu} = (v^0,\mathbf{v}=v^i)$ = vector $(4=4^1)$: $\boldsymbol{T} = \boldsymbol{T_{asym}} + \boldsymbol{T_{sym}} = T^{\mu\nu} = [[t^{00}, t^{0k}], [t^{j0}, t^{jk}]]$ = matrix $(16=4^2)$: etc.
Technically, these are all 4-Tensors = 4D Tensors; specify precisely using the #D (m,n)-Tensor notation {# dims, m $^{\text{upper}}$, n $_{\text{lower}}$ indices}
All SR 4-Tensors obey $T^{\mu1'\ldots\mu m'} = \Lambda^{\mu1'}{}_{\nu1}\,\Lambda^{\mu2'}{}_{\nu2} \ldots \Lambda^{\mu m'}{}_{\nu m}\,T^{\nu1\ldots\nu m}$: m is the # of indices and a separate Lorentz Transform Λ for each index

<Time·Space> 4-Vector Name matches its spatial 3-vector component name: ex. 4-Position $\mathbf{R}$ = (c*time t,3-position $\mathbf{r}$) [length]→[m]
LightSpeed Factor (c) will be in temporal component as required to make all [dimensional-units] of a 4-Vector's components match

SR 4-Vector V = (**4D SpaceTime 4-Vector**) = (1D temporal 3-scalar, **3D spatial 3-vector**) → 4D (1+3)-splitting into (v^t,v^x,v^y,v^z)

Tensor-index-notation in non-bold: ex. $A^{\mu} = (a^{\mu}) = (a^0,a^j) = (a^0,a^1,a^2,a^3)$: ex. $A^{\mu\nu} = [[a^{\mu\nu}]] = [[a^{00}, a^{0k}],[a^{j0}, a^{jk}]]$
4-Vectors (4D) in **bold** UPPERCASE: ex. $\mathbf{A} = \overline{\mathbf{A}} = (\mathbf{A}) = (a^0,\mathbf{a}) = (a^0,a^1,a^2,a^3)$
3-vectors (3D) in **bold** lowercase: ex. $\mathbf{a} = \overline{\mathbf{a}} = \vec{\mathbf{a}} = (\mathbf{a}) = (\mathbf{a}) = (a^1,a^2,a^3)$
Temporal scalars (1D) in non-bold, usually lowercase, 0^{th} component: ex. a^0 , a_0 "Count from 1, but index from 0 :-) "
Individual non-grouped components of 4-Tensors in non-bold: ex. $\mathbf{A} = (a^0,a^1,a^2,a^3) = (a^0,\mathbf{a})$ Vectors are grouped structures, thus **bold**
Rest scalars (invariants) in non-bold, denoted with naught ($_o$): ex. m_o : from $\mathbf{P} = m_o\mathbf{U}$ "A rest-frame is a valid relativistic concept"

Upper index 4-Vector $\mathbf{A} = \overline{\mathbf{A}} = A^{\mu} = (a^{\mu}) = (a^0,a^j)$: Lower index 4-CoVector $\underline{\mathbf{B}} = B_{\mu} = (b_{\mu}) = (b_0,b_j)$ a.k.a 4-DualVector=4D-OneForm
Index lowering/raising via Minkowski Metric $\boldsymbol{\eta}$: ex. $R_{\mu} = \eta_{\mu\nu}R^{\nu}$ or $\partial^{\mu} = \eta^{\mu\nu}\partial_{\nu}$ or $U^{\mu} = \eta^{\mu\nu}U_{\nu}$ with 4-Velocity $\mathbf{U} = U^{\mu} = \gamma(c,\mathbf{u}) = \gamma c(1,\boldsymbol{\beta})$

SR Relativistic Gamma $\gamma = 1/\sqrt{[1 - \boldsymbol{\beta}\cdot\boldsymbol{\beta}]}$: Relativistic $\boldsymbol{\beta} = \mathbf{u}/c = \{0..1\}\hat{\mathbf{n}}$: ProperTimeDerivative $(d/d\tau) = \gamma(d/dt) = (\mathbf{U}\cdot\boldsymbol{\partial})$

4D (1,0)-Tensor = 4-Vector: $\mathbf{A} = \overline{\mathbf{A}} = A^{\mu}$: ex. 4-Momentum $\mathbf{P} = P^{\mu} = (E/c,\mathbf{p}) = (mc,m\mathbf{u}) = m_o\mathbf{U} = m_o\gamma(c,\mathbf{u}) = m(c,\mathbf{u})$
4D (0,1)-Tensor = 4-CoVector = 4D-OneForm: $\underline{\mathbf{A}} = A_{\mu}$: ex. 4D GradientOneForm $\partial_{\mu} = (\partial_t/c,\nabla) = (\partial/\partial R^{\mu})$

"Unit"Temporal 4-Vector $\mathbf{T} = \gamma(1,\boldsymbol{\beta})$, with Lorentz Scalar Invariant $\mathbf{T}\cdot\mathbf{T} = T^{\mu}T_{\mu} = \gamma^2[1^2 - \boldsymbol{\beta}\cdot\boldsymbol{\beta}] = +1$ $\mathbf{T} = \mathbf{U}/c$
Null 4-Vector $\mathbf{N} \sim (\pm|\mathbf{a}|,\mathbf{a}) = a(\pm1,\hat{\mathbf{n}})$, with Lorentz Scalar Invariant $\mathbf{N}\cdot\mathbf{N} = N^{\mu}N_{\mu} = a^2[1^2 - \hat{\mathbf{n}}\cdot\hat{\mathbf{n}}] = 0$
"Unit"Spatial 4-Vector $\mathbf{S} = \gamma_{\beta\hat{n}}(\boldsymbol{\beta}\cdot\hat{\mathbf{n}},\hat{\mathbf{n}})$, with Lorentz Scalar Invariant $\mathbf{S}\cdot\mathbf{S} = S^{\mu}S_{\mu} = \gamma_{\beta\hat{n}}{}^2[(\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^2 - \hat{\mathbf{n}}\cdot\hat{\mathbf{n}}] = -1$ $\mathbf{T}\cdot\mathbf{S} = (\gamma^*\gamma_{\beta\hat{n}})[\boldsymbol{\beta}\cdot\hat{\mathbf{n}}-\boldsymbol{\beta}\cdot\hat{\mathbf{n}}] = 0$
$\mathbf{T}\cdot\mathbf{S} = 0 \leftrightarrow (\mathbf{T} \perp_{4D} \mathbf{S})$

Time-like separated <Events>	Null-like separated <Events>	Space-like separated <Events>
Invariant Temporal Causality=Time-ordering	Invariant Null LightCone	Invariant Spatial Topology=Space-ordering
Moving Clock = ←|Time Dilation|→	| |Invariant LightSpeed (c)| |	Moving Ruler = →|Length Contraction|←
Relativity of Stationarity = non-Topological	Causal & Topological	*Relativity* of Simultaneity = non-Causal

Alternate ways/styles of writing 4-Vector and 4-Tensor expressions in Physics:

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($\mathbf{A} \cdot \mathbf{B}$) is a 4-Vector style, which uses vector-notation {ex. **bold** vector-grouping, inner product "dot=·", exterior product "wedge=^"}, and can show relations very compactly. There are 4D analogs to the standard 3D vector rules, some of which are shown below:
Use **bold** UPPERCASE to represent 4-Vectors: $\mathbf{A} = (\mathbf{A}) = (a^0,\mathbf{a}) = (a^0,a^1,a^2,a^3)$ & **bold** lowercase for 3-vectors: $\mathbf{a} = (\mathbf{a}) = (a^1,a^2,a^3)$
4D Lorentz Scalar Product $(\mathbf{A} \cdot \mathbf{B}) = (A^\mu \eta_{\mu\nu} B^\nu) = (a^0b^0 - a^1b^1 - a^2b^2 - a^3b^3) = (a^0b^0 - \mathbf{a} \cdot \mathbf{b})$: { 4D $\cdot = \eta_{\mu\nu} = \mathrm{Diag}[+1,-1,-1,-1]_{(\mathrm{Cartesian})}$ }
3D vector dot product $(\mathbf{a} \cdot \mathbf{b}) = (a^j \delta_{jk} b^k) = (+a^1b^1 + a^2b^2 + a^3b^3) = +(\mathbf{a} \cdot \mathbf{b})$: { 3D $\cdot = \delta_{jk} = \mathrm{Diag}[+1,+1,+1]_{(\mathrm{Cartesian})}$ }
The 4-AngularMomentum uses the exterior wedge product (^) and can be written as $\boldsymbol{M} = \mathbf{R}\hat{}\mathbf{P} = [[0,-c\mathbf{n}],[c\mathbf{n}^T,\mathbf{l}=\mathbf{r}\hat{}\mathbf{p}]] = (R^\mu P^\nu - R^\nu P^\mu)$.
The 4D Gauss' Theorem in SR is: $\int_\Omega d^4\mathbf{X}\,(\partial\cdot\mathbf{V}) = \oint_{\partial\Omega} dS\,(\mathbf{V}\cdot\hat{\mathbf{N}})$: Divergence integral of vector field = Boundary-flux integral with:
Ω as a 4D simply-connected region of Minkowski SpaceTime
$\partial\Omega = S$ as its 3D boundary with its own 3D Volume element dS [m^3] and outward-pointing 4-Unit"HyperSurface"Normal $\hat{\mathbf{N}}$ [1] 3 DoF
$d^4\mathbf{X} = (c\,dt)(d^3\mathbf{x}) = (c\,dt)(dx\,dy\,dz)$ as a 4D Infinitesimal Volume Element [m^4], with 3D Volume = $\int d^3\mathbf{x} = \iiint(dx\,dy\,dz)$ [m^3]
$\mathbf{V} = \mathbf{V}(t,\mathbf{x})$ as an arbitrary 4-Vector field in SpaceTime [dim V], $\boldsymbol{\partial} = (\partial_t/c,-\nabla)$ is the 4-Gradient [1/m]

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$(A^\mu \eta_{\mu\nu} B^\nu)$ is a Ricci-Calculus style, which uses tensor-index-notation and is useful for more complicated expressions, especially to clarify expressions involving tensors with >1 index, ex. the Faraday EM Tensor $F^{\mu\nu} = (\partial^\mu A^\nu - \partial^\nu A^\mu) = [[0,-e^{0j}/c],[e^{i0}/c,-\varepsilon^{ij}{}_k b^k]] = \boldsymbol{\partial}\hat{}\mathbf{A}$.
Tensor notation and rules include concepts such as:
Kronecker Delta $(\delta^\mu{}_\nu) = \mathrm{Diagonal}[+1,+1,+1,+1] = \mathrm{Diagonal}[+1,+I_{[3]}] = \mathrm{Diagonal}[+1,+\delta^j{}_k]$ = 4D Identity = $I_{[4]} = \eta^\mu{}_\nu = g^\mu{}_\nu$
Levi-Civita Symbol $(\varepsilon^{ij}{}_k)$ = {+1 for even permutation, -1 for odd permutation, else 0} = Totally Anti-Symmetric Tensor, also $(\varepsilon^{\mu\nu}{}_{\rho\sigma})$ etc.
Index lowering/raising with a Metric $\boldsymbol{g} \to g_{\mu\nu}$ or $g^{\mu\nu}$: typically using limit-case $\boldsymbol{g} \to \boldsymbol{\eta}$ for "flat" SR Minkowski Metric $\eta_{\mu\nu}$ or $\eta^{\mu\nu}$
Einstein summation convention {paired upper/lower indices are summed over} ex. $A^\mu B_\mu = a^0b_0+a^1b_1+a^2b_2+a^3b_3 = (A^\mu \eta_{\mu\nu} B^\nu) = A_\nu B^\nu$
Symmetric:AntiSymmetric Tensor decomposition $\{T^{\mu\nu} = S^{\mu\nu} + A^{\mu\nu}\}$, with $S^{\mu\nu} = (T^{\mu\nu}+T^{\nu\mu})/2$ and $A^{\mu\nu} = (T^{\mu\nu}-T^{\nu\mu})/2$
Tensor Contraction of Symmetric with AntiSymmetric yields zero $\{S^{\mu\nu}A_{\mu\nu} = 0\}$, from $\{ S^{\mu\nu} = +S^{\nu\mu} \}$ and $\{ A^{\mu\nu} = -A^{\nu\mu} \}$ parts of $\{ T^{\mu\nu} \}$
Proof: 10 components + 6 components = 16 comps
$S^{\mu\nu}A_{\mu\nu}$ = {swaping dummy indices}$\to S^{\nu\mu}A_{\nu\mu} = (S^{\nu\mu})(A_{\nu\mu}) = (+S^{\mu\nu})(A_{\nu\mu}) = (+S^{\mu\nu})(-A_{\mu\nu}) = -S^{\mu\nu}A_{\mu\nu} = 0$, since { C = -C = 0 }
The Symmetric Tensor is further decomposed into an Isotropic part $S_{iso}{}^{\mu\nu} = (S^\alpha{}_\alpha/4)\eta^{\mu\nu}$ and zero-trace Anisotropic part $S_{aniso}{}^{\mu\nu} = S^{\mu\nu} - S_{iso}{}^{\mu\nu}$
So, $\{T^{\mu\nu} = S_{iso}{}^{\mu\nu} + S_{aniso}{}^{\mu\nu} + A^{\mu\nu}\}$ This is manifestly invariant: The Poincaré Group Symmetry operations respect these decompositions, meaning that boosts, rotations, etc. don't intermix them, unlike the (temporal+mixed+spatial)-splittings, which can get intermixed.

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This paper uses a mix of the two styles, as both are useful in various circumstances. ex. The following are all equivalent:
$(\mathbf{A} \cdot \mathbf{B}) = (A^\mu \eta_{\mu\nu} B^\nu) = (a^0b^0 - a^1b^1 - a^2b^2 - a^3b^3) = (a^0b^0 - a^j \delta_{jk} b^k) = (a^0b^0 - \mathbf{a} \cdot \mathbf{b}) = (a^0{}_o b^0{}_o)$
$= A^\mu B_\mu = (a^0b_0 + a^1b_1 + a^2b_2 + a^3b_3)$ = Einstein Summation
$= A_\nu B^\nu = (a_0b^0 + a_1b^1 + a_2b^2 + a_3b^3)$ = Einstein Summation
This leads to the interesting observations that the Minkowski Metric $(\eta_{\mu\nu})$ can contract with either side $\{ A^\mu B_\mu = (A^\mu \eta_{\mu\nu} B^\nu) = A_\nu B^\nu \}$ and that it formally acts like a 4D vector dot operator $(\eta_{\mu\nu} = \cdot)$.

Notes on Poincaré SpaceTime Group, Tensor Linear Mapping, Lie Group, etc.:
SR 4-Vectors have a Poincaré Group linear mapping $(V^{\mu'} = \Lambda^{\mu'}{}_\nu V^\nu + \Delta V[\Delta X^{\mu'}])$ which preserves interval-magnitude: $(V^{\mu'}V_{\mu'} = V^\nu V_\nu)$.

For a given mapping, the Lorentz-Transform $(\Lambda^{\mu'}{}_\nu)$ [dimensionless] & SpaceTime-Translation-Transform $(\Delta X^{\mu'})$ [m] are constants.
Since the SpaceTime-Translation part is not dimensionless, one has to take care handling it. Temporarily set multiplier $\{ \Lambda^{\mu'}{}_\nu \to \delta^{\mu'}{}_\nu \}$

4-Position $X^{\mu'} = \Lambda^{\mu'}{}_\nu X^\nu + \Delta X^{\mu'} = X^{\mu'} + \Delta X^{\mu'} \to$ Not SpaceTime Translation Invariant, hence not Poincaré Invariant $\{\Delta X^{\mu'} \to 0^{\mu'}\}$

4-Displacement $\Delta X_{21}{}^{\mu'} = X_2{}^{\mu'} - X_1{}^{\mu'} = (\Lambda^{\mu'}{}_\nu X_2{}^\nu + \Delta X^{\mu'}) - (\Lambda^{\mu'}{}_\nu X_1{}^\nu + \Delta X^{\mu'}) = (X_2{}^{\mu'}) - (X_1{}^{\mu'}) \to$ Poincaré Invariant$\{\Delta X^{\mu'}$ is unrestricted$\}$

4-Velocity $U^{\mu'} = (d/d\tau)[\Lambda^{\mu'}{}_\nu X^\nu + \Delta X^{\mu'}] = (d/d\tau)[X^{\mu'}] + (d/d\tau)[\Delta X^{\mu'}] = U^{\mu'} + 0^{\mu'} = U^{\mu'} \to$ Poincaré Invariant$\{\Delta X^{\mu'}$ is unrestricted$\}$

Likewise, any 4-Vector that is based on 4-Velocity {many, since 4-Velocity only has 3 independent components, it can be multiplied by a Lorentz Scalar to make a new 4-Vector} will be Poincaré Invariant. Basically, only the 4-Position is not, although it is still Lorentz Invariant. Consider 4-Position **R** as a 4-Displacement Δ**R** in which one of the endpoints is "pinned" to the 4-Origin $\mathbf{O} = (0,\mathbf{0})$. Since it is "pinned", it can't be SpaceTime-Translated, but it can still be Lorentz-Transform "Rotated" about the 4-Origin.

[↻] [→]
The Poincaré Group is a Lie Group, and can be written as a Unitary Operation: $U(\Lambda^{\mu'}{}_\nu , \Delta X^{\mu'}) = e\hat{}[(i/2\hbar)\omega_{\mu\nu}M^{\mu\nu}]\, e\hat{}[(i/\hbar)\Delta X_\mu P^\mu]$ with:
4-LinearMomentum P^μ [kg·m/s] as generator of SpaceTime-Translation-Transforms with ΔX_μ [m] encoding the 1+3=4 displacements
4-AngularMomentum M^μ [kg·m^2/s] as generator of Lorentz-Transforms with anti-symmetric $\omega_{\mu\nu}$ [1] encoding the 3 angles + 3 boosts
$[\Delta X_\mu P^\mu]$ and $[\omega_{\mu\nu}M^{\mu\nu}]$ and $(\hbar)$ all have dimensional-units of [Action = kg·m^2/s = J·s] : Infinitesimal $\Lambda^{\mu'}{}_\nu = \delta^{\mu'}{}_\nu + \omega^{\mu'}{}_\nu + \ldots$

The following 4-Vectors { 4D (1,0)-Tensors } are all elements of classical SR and EM:

4-Position	$\mathbf{R} = R^\mu = (ct,\mathbf{r})$	[m]	Alt. $\mathbf{X} = X^\mu = (ct,\mathbf{x})$ only Lorentz, not Poincaré Invariant
4-Displacement	$\Delta\mathbf{R} = \Delta R^\mu = (c\Delta t,\Delta\mathbf{r})$	[m]	Finite $\Delta\mathbf{R} = \mathbf{R}_2 - \mathbf{R}_1$ fully Poincaré Invariant
4-Differential	$d\mathbf{R} = dR^\mu = (cdt,d\mathbf{r})$	[m]	Infinitesimal $d\mathbf{R}$ fully Poincaré Invariant
4-Velocity	$\mathbf{U} = U^\mu = \gamma(c,\mathbf{u})$	[m/s]	$\gamma = 1/\sqrt{[1 - \boldsymbol{\beta}\cdot\boldsymbol{\beta}]}$ [1] : $\boldsymbol{\beta} = \mathbf{u}/c$ [1]
4-Momentum	$\mathbf{P} = P^\mu = (E/c,\mathbf{p})$	[kg·m/s = N·s]	Invariant SR Action $S = -(\mathbf{P}\cdot\mathbf{X})$ [kg·m²/s]
4-WaveVector	$\mathbf{K} = K^\mu = (\omega/c,\mathbf{k})$	[{rad}/m]	Invariant SR WavePhase $\Phi = -(\mathbf{K}\cdot\mathbf{X})$ [{rad}]
4-Gradient	$\partial = \partial^\mu = (\partial_t/c,-\nabla)$	[1/m]	$\partial=\partial_{\mathbf{R}}$: See also 4-VelocityGradient $\partial_{\mathbf{U}}$ [s/m]
4-(Dust)NumberFlux	$\mathbf{N} = N^\mu = (nc,\mathbf{n})$	[#/(m²·s) = (#/m³)·(m/s)]	
4-Current(Density)=4-ChargeFlux	$\mathbf{J} = J^\mu = (\rho c,\mathbf{j})$	[C/(m²·s) = (C/m³)·(m/s) = A/m²]	
4-(EM)VectorPotential	$\mathbf{A} = A^\mu = (\varphi/c,\mathbf{a})$	[kg·m/(C·s) = T·m]	Alt. $\mathbf{A}_{EM} = A_{EM}^\mu = (\varphi_{EM}/c,\mathbf{a}_{EM})$
4-(Minkowski)Force	$\mathbf{F} = F^\mu = \gamma(\dot{E}/c,\mathbf{f})$	[kg·m/s² = N]	

The mathematical Lorentz Scalar Product of two generic 4-Vectors $\mathbf{A} = A^\mu = (a^0,\mathbf{a})$ and $\mathbf{B} = B^\mu = (b^0,\mathbf{b})$ is:

$(\mathbf{A}\cdot\mathbf{B}) = A^\mu\eta_{\mu\nu}B^\nu = A_\nu B^\nu = A^\mu B_\mu = (a^0b^0 - \mathbf{a}\cdot\mathbf{b}) = (a^0{}_o b^0{}_o)$ which is a Lorentz Invariant "Rest" 4-Scalar { 4D (0,0)-Tensor }

Applying this rule to various SR 4-Vectors gives the following SR Lorentz Invariants (rest values indicated with naught $_o$):

$(\mathbf{R}\cdot\mathbf{R}) = (ct)^2 - \mathbf{r}\cdot\mathbf{r} = (ct_o)^2 = (c\tau)^2 = (i\lvert\mathbf{r}_o\rvert)^2$	: Proper Time ($t_o = \tau$), Proper Length ($\lvert\mathbf{r}_o\rvert$)	[s], [m]	{conversion factor c}
$(\mathbf{U}\cdot\mathbf{U}) = \gamma^2[c^2 - \mathbf{u}\cdot\mathbf{u}] = c^2$	: Invariant LightSpeed (c)	[m/s]	
$(\mathbf{P}\cdot\mathbf{P}) = (E/c)^2 - \mathbf{p}\cdot\mathbf{p} = (E_o/c)^2 = (m_oc)^2$	: Rest Energy (E_o), Rest Mass (m_o)	[kg·m²/s² = J], [kg]	{conversion factor c²}
$(\mathbf{K}\cdot\mathbf{K}) = (\omega/c)^2 - \mathbf{k}\cdot\mathbf{k} = (\omega_o/c)^2$	: Rest Angular Frequency (ω_o)	[{rad}/s]	
$(\partial\cdot\partial) = (\partial_t/c)^2 - \nabla\cdot\nabla = (\partial_{to}/c)^2 = (\partial_\tau/c)^2$	: Proper Time Partial ($\partial_\tau=\partial_{to}$)	[1/s]	d'Alembertian 4D Wave
$(\mathbf{N}\cdot\mathbf{N}) = (nc)^2 - \mathbf{n}\cdot\mathbf{n} = (n_oc)^2$	: Rest Number Density (n_o)	[#/m³]	Equation $(\partial\cdot\partial)$ [1/m²]
$(\mathbf{J}\cdot\mathbf{J}) = (\rho c)^2 - \mathbf{j}\cdot\mathbf{j} = (\rho_oc)^2$	: Rest Charge Density (ρ_o)	[C/m³]	
$(\mathbf{A}\cdot\mathbf{A}) = (\varphi/c)^2 - \mathbf{a}\cdot\mathbf{a} = (\varphi_o/c)^2$	: Rest Electric:Scalar Potential (φ_o)	[kg·m²/(C·s²) = V = J/C]	
$(\mathbf{F}\cdot\mathbf{F}) = \gamma^2[(\dot{E}/c)^2 - \mathbf{f}\cdot\mathbf{f}] = (\dot{E}_o/c)^2$	: Rest Power ($\dot{E}_o$)	[kg·m²/s³ = W = J/s]	
$(\mathbf{U}\cdot\partial) = \gamma[\partial_t + \mathbf{u}\cdot\nabla] = \gamma d/dt = d/d\tau$	: Proper Time Derivative ($d/d\tau$)	[1/s]	
$(\mathbf{K}\cdot\mathbf{R}) = (\omega t) - \mathbf{k}\cdot\mathbf{r} = (\omega_ot_o) = -\Phi$	: Invariant SR WavePhase (Φ) or (Φ_{phase})	[{rad}]	
$(\mathbf{P}\cdot\mathbf{R}) = (Et) - \mathbf{p}\cdot\mathbf{r} = (E_ot_o) = -S$	: Invariant SR Action (S) or (S_{action})	[kg·m²/s = J·s = Action]	

The SR 4-Vectors have some fundamental relations between one another (again using rest naught $_o$ = proper = invariant notation):

4-Position	$\mathbf{R} = R^\mu = (ct,\mathbf{r}) \in$ <Event> $\in$ 4D <Time·Space>	with 4 = (1+3)-splitting, $R_\mu = \eta_{\mu\nu}R^\nu$
4-Velocity	$\mathbf{U} = U^\mu = \gamma(c,\mathbf{u}=\dot{\mathbf{r}}) = (\mathbf{U}\cdot\partial)\mathbf{R} = (d/d\tau)\mathbf{R} = d\mathbf{R}/d\tau = \gamma d\mathbf{R}/dt$	$\mathbf{u} = \dot{\mathbf{r}} = d\mathbf{r}/dt$: $(\mathbf{U}\cdot\partial)=(d/d\tau)=\gamma(d/dt)$
4-Momentum	$\mathbf{P} = P^\mu = (E/c=mc,\mathbf{p}=m\mathbf{u}) = (E_o/c^2)\mathbf{U} = m_o\mathbf{U} = \gamma m_o(c,\mathbf{u})$	$E = mc^2 = \gamma E_o = \gamma m_oc^2$: $E_o = m_oc^2$
4-WaveVector	$\mathbf{K} = K^\mu = (\omega/c=1/c\text{Ŧ},\mathbf{k}=\omega\hat{\mathbf{n}}/v_{phase}=\hat{\mathbf{n}}/ƛ=\omega\mathbf{u}/c^2) = (\omega_o/c^2)\mathbf{U}$	$\hat{\mathbf{n}}$ = unit-direction 3-vector {2 DoF}
4-Gradient	$\partial = \partial^\mu = (\partial_t/c,-\nabla) = (\partial/\partial R_\mu)$: with $\partial_\mu = (\partial_t/c,\nabla) = (\partial/\partial R^\mu)$	$\partial_t = \partial/\partial t$: $\nabla = (\partial_x,\partial_y,\partial_z) = (\partial/\partial x,\partial/\partial y,\partial/\partial z)$
4-(Dust)NumberFlux	$\mathbf{N} = N^\mu = (nc,\mathbf{n}=n\mathbf{u}) = n_o\mathbf{U} = \gamma n_o(c,\mathbf{u})$	allows "fluid:density" concept in SR
4-Current(Density)=4-ChargeFlux	$\mathbf{J} = J^\mu = (\rho c,\mathbf{j}=\rho\mathbf{u}) = \rho_o\mathbf{U} = \gamma\rho_o(c,\mathbf{u}) = q\gamma n_o(c,\mathbf{u}) = q\mathbf{N}$	ρ = EM Charge Density
4-(EM)VectorPotential	$\mathbf{A} = A^\mu = (\varphi/c,\mathbf{a}=\varphi\mathbf{u}/c^2) = (\varphi_o/c^2)\mathbf{U} = (\gamma\varphi_o/c^2)(c,\mathbf{u})$	φ = EM Scalar Potential = Voltage
4-(Minkowski)Force	$\mathbf{F} = F^\mu = \gamma(\dot{E}/c,\mathbf{f}=\dot{\mathbf{p}}) = (\mathbf{U}\cdot\partial)\mathbf{P} = (d/d\tau)\mathbf{P} = d\mathbf{P}/d\tau = \gamma d\mathbf{P}/dt$	$\dot{E} = dE/dt$, $\mathbf{f} = \dot{\mathbf{p}} = d\mathbf{p}/dt$
Faraday EM 4-Tensor	$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha = (\partial\wedge\mathbf{A})$	$F^{\alpha\beta}$ = AntiSymmetric 4D(2,0)-Tensor
Maxwell Equation with Source	$\partial\cdot F^{\alpha\beta} = (\partial\cdot\partial)\mathbf{A} - \partial(\partial\cdot\mathbf{A}) = (\mu_o)\mathbf{J}$	{or $(\partial\cdot\partial)\mathbf{A} = (\mu_o)\mathbf{J}$ in Lorenz-Gauge $(\partial\cdot\mathbf{A}) = 0$}
4D Lorentz Force Equation	$\mathbf{U}\cdot F^{\alpha\beta} = (1/q)\mathbf{F} = (1/q)d\mathbf{P}/d\tau$	q = EM charge
Conservation of EM Current(Density)	$\partial\cdot\mathbf{J} = 0$	$\partial_\alpha\partial_\beta F^{\alpha\beta} = 0 = (\mu_o)\partial\cdot\mathbf{J}$

Beautiful proof based on Symmetric:AntiSymmetric tensors

Fundamental Constants = 4D Lorentz Invariants = { 4D (0,0)-Tensors }which elegantly appear from SR 4-Vector formalism:

Light Speed (Vacuum)	(c)	[m/s]	v=c/n, with n = material index-of-refraction
Rest Mass = Invariant Mass = Proper Mass	(m_o)	[kg]	Varies depending on particle type
EM charge (=e for electron)	(q)	[C]	Varies depending on particle type
Electric Constant=Permittivity (Vacuum)	(ε_o)	[F/m = C²·s²/kg·m³]	$(\varepsilon_o\mu_o) = 1/c^2$
Magnetic Constant=Permeability (Vacuum)	(μ_o)	[H/m = kg·m/C²]	$(\varepsilon_o\mu_o) = 1/c^2$
Planck's Reduced = Dirac's Constant	($\hbar$)	[J·s = kg·m²/s = Action]/{rad}	We shall see this in the next section…
Boltzmann's Constant	(k_B)	[J/°K = kg·m²/°K·s²]	See the full presentation SRQM-RoadMap

Constants with (Vacuum) are considered in their non-interacting state, their effective values change with matter-interaction.

<Photon speed> varies in medium (v = c/n due to atomic interaction times), ex. going through a prism, which causes refraction effects.

Note: (c) is large, but never → ∞; ($\hbar$) is small, but never → 0; Always use realistic limits, ex. { $\lvert\mathbf{v}\rvert \ll c$ } for SR→CM, RQM→ QM

One can divide two generic 4-Vectors $\mathbf{A} = A^\mu = (a^0,\mathbf{a})$ and $\mathbf{B} = B^\mu = (b^0,\mathbf{b})$ by using an intermediary 4-Vector $\mathbf{V} = V^\mu = (v^0,\mathbf{v})$:
$|\mathbf{A}|/|\mathbf{B}| = (\mathbf{A}\cdot\mathbf{V})/(\mathbf{B}\cdot\mathbf{V}) = (a^0{}_o v^0{}_o)/(b^0{}_o v^0{}_o) = (a^0{}_o/b^0{}_o)$ which is a Lorentz Invariant 4-Scalar { 4D (0,0)-Tensor }

Applying the division rule with certain SR 4-Vector combinations leads to the following very "suggestive" relations:
$(\mathbf{P}\cdot\mathbf{U})/(\mathbf{K}\cdot\mathbf{U}) = \gamma(E - \mathbf{p}\cdot\mathbf{u})/[\gamma(\omega - \mathbf{k}\cdot\mathbf{u})] = E_o/\omega_o$ $\rightarrow |\mathbf{P}|/|\mathbf{K}| = E_o/\omega_o$
$(\mathbf{P}\cdot\mathbf{K})/(\mathbf{K}\cdot\mathbf{K}) = (E\omega/c^2 - \mathbf{p}\cdot\mathbf{k})/[(\omega/c)^2 - \mathbf{k}\cdot\mathbf{k}] = m_o\omega_o/(\omega_o/c)^2 = m_oc^2/\omega_o$ $\rightarrow |\mathbf{P}|/|\mathbf{K}| = E_o/\omega_o$
$(\mathbf{P}\cdot\mathbf{P})/(\mathbf{K}\cdot\mathbf{P}) = (E^2/c^2 - \mathbf{p}\cdot\mathbf{p})/(E\omega/c^2 - \mathbf{p}\cdot\mathbf{k}) = (m_oc)^2/(m_o\omega_o) = m_oc^2/\omega_o$ $\rightarrow |\mathbf{P}|/|\mathbf{K}| = E_o/\omega_o$
$(\mathbf{P}\cdot\mathbf{R})/(\mathbf{K}\cdot\mathbf{R}) = (Et - \mathbf{p}\cdot\mathbf{r})/(\omega t - \mathbf{k}\cdot\mathbf{r}) = (-S_{\text{action,free particle}})/(-\Phi_{\text{phase,planewave}}) = (E_ot_o)/(\omega_ot_o)$ $\rightarrow |\mathbf{P}|/|\mathbf{K}| = E_o/\omega_o$
Also, from the definitions:
$\mathbf{K} = (\omega_o/c^2)\mathbf{U}$ or $\mathbf{U} = (c^2/\omega_o)\mathbf{K}$
Thus:
$\mathbf{P} = (E_o/c^2)\mathbf{U} = (E_o/c^2)(c^2/\omega_o)\mathbf{K} = (E_o/\omega_o)\mathbf{K}$: ⤞ $\mathbf{P} = (E_o/\omega_o)\mathbf{K}$ ⇝ $\rightarrow |\mathbf{P}|/|\mathbf{K}| = E_o/\omega_o$

Analysis of Dirac's Constant ($\hbar = h/2\pi$) : Planck's Constant ($h = 2\pi\hbar$), and (E_o/ω_o) in the context of SRQM:
It is an empirical (observational) fact that the Lorentz Scalar Invariant $(E_o/\omega_o) = (\gamma E_o/\gamma\omega_o) = (E/\omega)$ ==> $(\hbar)$ [J·s]/{rad}
for all known experimental measurements. The SR 4D-Tensor rules show that one doesn't need a quantum axiom for this.
$(\hbar)$ is actually an empirically-measurable quantity, just like (c), (e), (G), (k_B), (μ_o), (ε_o) or the other fundamental constants, which are also 4D Lorentz Scalar Invariants. $(\hbar)$ can be measured classically {without need of quantum axioms} from the photoelectric effect, from the inverse photoelectric effect, from LED's (injection electroluminescence), from Atomic Line Spectra (Bohr Atom), from the Duane-Hunt Law in Bremsstrahlung, from Electron Diffraction in crystals, from the Watt/Kibble-Balance, from the Sagnac Effect, from Incandescent Blackbody Intensity-Temperature Relations, from Compton Scattering, from the Stern-Gerlach experiment, etc.

For physics simulations of measurements of Dirac's : Planck's Constant $(\hbar : h)$, see:
http://scirealm.org/Physics-PlanckConstantViaAtomicLineSpectra.html (transitions of electron energy-levels)
http://scirealm.org/Physics-PlanckConstantViaComptonScattering.html (photon:electron collisions)
http://scirealm.org/Physics-PlanckConstantViaElectronDiffraction.html (electron:atomic crystal scattering)
http://scirealm.org/Physics-PlanckConstantViaGravityInducedInterferometry.html (gravity interaction)
http://scirealm.org/Physics-PlanckConstantViaIncandescence.html (temperature interaction)
http://scirealm.org/Physics-PlanckConstantViaLEDs.html (electron:photon interaction)
http://scirealm.org/Physics-PlanckConstantViaSagnacEffect.html (rotation interaction)
http://scirealm.org/Physics-PlanckConstantViaSternGerlach.html (magnetic-field interaction)

$\{E_o/\omega_o = E/\omega = \hbar\}$ implies the following SR (wave:particle-like ⦚ ·) relation:
$\mathbf{P} = \hbar\mathbf{K} = (E/c,\mathbf{p}) = \hbar(\omega/c,\mathbf{k}) = (\hbar\omega/c,\hbar\mathbf{k})$
The temporal part $\{E = \hbar\omega\}$ gives Einstein's photoelectric quantum "postulate". Again, emphasis: derived from SR.
The spatial part $\{\mathbf{p} = \hbar\mathbf{k}\}$ gives de Broglie's matter-wave quantum "postulate". Again, emphasis: derived from SR.

This is very similar to Einstein's other SR (particle-like ·) relation: {both are simple Lorentz 4-Scalar times SR 4-Vector}
$\mathbf{P} = m_o\mathbf{U} = (E/c,\mathbf{p}) = m_o\gamma(c,\mathbf{u}) = m(c,\mathbf{u}) = (mc,m\mathbf{u}) = (E/c,E\mathbf{u}/c^2) = (E/c^2)(c,\mathbf{u}) = (E_o/c^2)\gamma(c,\mathbf{u}) = (E_o/c^2)\mathbf{U}$
The temporal part $\{E = \gamma m_oc^2 = mc^2 = \gamma E_o\}$ gives Einstein's famous energy:mass relation (in both rest $_o$ & relativistic γ forms).
The spatial part $\{\mathbf{p} = \gamma m_o\mathbf{u} = m\mathbf{u} = E\mathbf{u}/c^2 = \gamma E_o\mathbf{u}/c^2\}$ gives Einstein's relativistic momentum.

The 4-WaveVector $\mathbf{K}$ exists in all physics/mathematical contexts as a solution of the 4D Invariant d'Alembertian $(\boldsymbol{\partial}\cdot\boldsymbol{\partial}) = (\partial_t/c)^2 - \nabla\cdot\nabla$:
It is an empirical & mathematical fact that all waves (classical/relativistic/EM/quantum/purely-mathematical) can be modeled using complex planewaves which obey a principle of superposition: "General" Wave $\Psi = \Sigma_n [\psi_n]$ = sum of complex planewaves.
An individual wavefunction of form $\psi = (a)e^{\pm i(\mathbf{K}\cdot\mathbf{X})} = (a)e^{\mp i\Phi} = (a)e^{\mp iS_{action}/\hbar}$ has tensor amplitude (a) that can be:

- 4D (0,0)-Tensor A {ex. Quantum Scalar}
- 4D (1,0)-Tensor A^μ {ex. EM/Photonic/Proca}
- 4D (2,0)-Tensor $A^{\mu\nu}$ {ex. Gravitational Wave}

This gives the mathematical 4-Vector relation: So, all waves have an amplitude tensor $A^{\cdots}$ and propagation tensor $\mathbf{K}$
$\boldsymbol{\partial} = -i\mathbf{K} = (\partial_t/c,-\nabla) = -i(\omega/c,\mathbf{k})$
The temporal part $\{\partial_t = -i\omega\}$ or $\{\omega = i\partial_t\}$ gives temporal:frequency complex planewave change:operator
The spatial part $\{\nabla = i\mathbf{k}\}$ or $\{\mathbf{k} = -i\nabla\}$ gives spatial:wavenumber complex planewave change:operator

$\psi = (a)e^{\pm i(\mathbf{K}\cdot\mathbf{X})}$; There exists also $\psi^* = (a^*)e^{\mp i(\mathbf{K}\cdot\mathbf{X})}$, giving $\psi^*\psi = (a^*)(a) = |\psi|^2$, independent of the phase part $\Phi = -(\mathbf{K}\cdot\mathbf{X})$
$\boldsymbol{\partial}[\psi] = \boldsymbol{\partial}[(a)e^{\pm i(\mathbf{K}\cdot\mathbf{X})}] = \pm i\mathbf{K}[(a)e^{\pm i(\mathbf{K}\cdot\mathbf{X})}] = \pm i\mathbf{K}[\psi]$, with the minus sign $\{\boldsymbol{\partial} = -i\mathbf{K}\}$ typically chosen for historical reasons.
$\boldsymbol{\partial}[\psi^*\psi] = (\boldsymbol{\partial}[\psi^*]\psi + \psi^*\boldsymbol{\partial}[\psi]) = (\mp i\mathbf{K}\psi^*\psi \pm i\mathbf{K}\psi^*\psi) = 0 = \boldsymbol{\partial}[(a^*)(a)]$, giving conservation of probability $\{\psi^*\psi = |\psi|^2\}$.

SR Wave Energy-Momentum, Dispersion, & Velocity Relations using 4D Tensors (esp. 4-Vectors):

4-Vector = 4D (1,0)-Tensor **Properties**			
4-Position	$\mathbf{R} = R^\mu = (ct,\mathbf{r})$	[m]	c = Invariant LightSpeed
4-Velocity	$\mathbf{U} = U^\mu = \gamma(c,\mathbf{u})$	[m/s]	$\gamma = 1/\sqrt{[1 - (u/c)^2]}$, $\mathbf{u}$ = particle:group velocity
4-Momentum	$\mathbf{P} = P^\mu = (E/c,\mathbf{p})$	[kg·m/s = N·s]	E = relativistic energy, $\mathbf{p}$ = relativistic 3-momentum
4-WaveVector	$\mathbf{K} = K^\mu = (\omega/c,\mathbf{k} = \omega\hat{\mathbf{n}}/v_{phase})$	[{rad}/m]	$v_{phase} = \omega/k$ = phase velocity

4-Vector = 4D (1,0)-Tensor **Relations**		
4-Velocity	$\mathbf{U} = d\mathbf{R}/d\tau$	τ = Proper Time, $d/d\tau$ = Proper Time Derivative
4-Momentum	$\mathbf{P} = (E_o/c^2)\mathbf{U} = m_o\mathbf{U} = \hbar\mathbf{K}$	E_o = Rest Energy, m_o = Rest mass, $\hbar$ = Dirac's const
4-WaveVector	$\mathbf{K} = (\omega_o/c^2)\mathbf{U}$	ω_o = Rest Angular Frequency

$\mathbf{K} = (\omega/c,\mathbf{k} = \omega\hat{\mathbf{n}}/v_{phase}) = (\omega_o/c^2)\mathbf{U} = (\omega_o/c^2)\gamma(c,\mathbf{u}) = (\gamma\omega_o/c^2)(c,\mathbf{u}) = (\gamma\omega_o/c,\gamma\omega_o\mathbf{u}/c^2)$ wave-motion [⦚] ⟿

The temporal part: $\omega = \gamma\omega_o$ {Relativistic ω : Rest ω_o} ang.-frequency

The spatial part: $\mathbf{k} = \omega\hat{\mathbf{n}}/v_{phase} = \gamma\omega_o\mathbf{u}/c^2 = \omega\mathbf{u}/c^2$ {Relativistic $\mathbf{k}$} 3-wavenumber

$\hat{\mathbf{n}}/v_{phase} = \mathbf{u}/c^2$

$\mathbf{u} = c^2\hat{\mathbf{n}}/v_{phase}$

$|v_{phase} * u| = c^2$

$\mathbf{P} = (E/c,\mathbf{p}) = (E_o/c^2)\mathbf{U} = (E_o/c^2)\gamma(c,\mathbf{u}) = (\gamma E_o/c^2)(c,\mathbf{u}) = (\gamma E_o/c,\gamma E_o\mathbf{u}/c^2)$ particle-motion [·] ⊸

The temporal part: $E = \gamma E_o$ {Relativistic E : Rest E_o} energy

The spatial part: $\mathbf{p} = \gamma E_o\mathbf{u}/c^2 = E\mathbf{u}/c^2 = Ec^2\hat{\mathbf{n}}/(c^2\ v_{phase}) = E\hat{\mathbf{n}}/v_{phase}$ {Relativistic $\mathbf{p}$} 3-momentum

$= \gamma m_o\mathbf{u} \quad = m\mathbf{u}$

$\mathbf{P} = (E/c,\mathbf{p}) = \hbar\mathbf{K} = \hbar(\omega/c,\mathbf{k} = \omega\hat{\mathbf{n}}/v_{phase})$ particle-wave duality [·⦚]

The temporal part: $E = \hbar\omega$ The Einstein photoelectric eqn.

The spatial part: $\mathbf{p} = \hbar\mathbf{k} = \hbar\omega\hat{\mathbf{n}}/v_{phase} = E\hat{\mathbf{n}}/v_{phase}$ The de Broglie matter-wave eqn.

So, 3-momentum $|\mathbf{p}|$ = Energy E / phase-velocity v_{phase} in general. Interesting note: Matter at-rest $\{\mathbf{p} = \mathbf{0}\}$ has $\{v_{phase} = \infty\}$

$(\mathbf{U}\cdot\mathbf{U}) = \gamma^2[(c)^2 - \mathbf{u}\cdot\mathbf{u}] = (c)^2\gamma^2[1 - \mathbf{u}\cdot\mathbf{u}/c^2] = (c)^2$

$(\mathbf{P}\cdot\mathbf{P}) = [(E/c)^2 - \mathbf{p}\cdot\mathbf{p}] = (E_o/c)^2 = (m_oc)^2$

$(E/c)^2 = \mathbf{p}\cdot\mathbf{p} + (E_o/c)^2$

$(E)^2 = \mathbf{p}\cdot\mathbf{p}c^2 + (E_o)^2$

$E^2 = (pc)^2 + (E_o)^2$

$E = \sqrt{[(pc)^2 + (E_o)^2]}$: If photonic, $(E_o=0)$, then $E_{photon} = |\mathbf{p}|c$, which gives $u_{photon} = v_{phase,photon} = c$, from $|v_{phase} * u| = c^2$

$(\mathbf{K}\cdot\mathbf{K}) = [(\omega/c)^2 - \mathbf{k}\cdot\mathbf{k}] = (\omega_o/c)^2$

$(\omega/c)^2 = k^2 + (\omega_o/c)^2$

$d[(\omega/c)^2] = d[k^2 + (\omega_o/c)^2]$

$2\omega\ d\omega/c^2 = 2k\ dk + 0$

$\omega\ d\omega/c^2 = k\ dk$

$d\omega/dk = c^2 k/\omega = c^2/v_{phase} = u$

Recapping:

$\omega/k = v_{phase}$	From the formal definition of 4-WaveVector $\mathbf{K} = K^\mu = (\omega/c,\mathbf{k} = \omega\hat{\mathbf{n}}/v_{phase})$
$d\omega/dk = u = v_{group} = v_{particle}$	Derived from Lorentz Scalar Product $(\mathbf{K}\cdot\mathbf{K}) = [(\omega/c)^2 - \mathbf{k}\cdot\mathbf{k}] = (\omega_o/c)^2$
$\mathbf{u} = c^2\hat{\mathbf{n}}/v_{phase}$	The relation between particle and wave velocities, from $\mathbf{K} = (\omega_o/c^2)\mathbf{U}$

One of the main points here is that many physics relations are easily derivable from simple 4-Vector tensorial rules and that many of these physics formulas are really just the 1D temporal & 3D spatial parts of these 4D <Time·Space> relations.

SR derived Sagnac Effect, Measuring Absolute Spatial Rotation, using 4D Tensors (esp. 4-Vectors):

4-Vector = 4D (1,0)-Tensor	**Properties**		
4-Position	$\mathbf{R} = R^\mu = (ct,\mathbf{r})$	[m]	c = Invariant LightSpeed
4-Differential	$d\mathbf{R} = dR^\mu = (cdt,d\mathbf{r})$	[m]	Calculus works normally on 4-Vectors
4-Velocity	$\mathbf{U} = U^\mu = \gamma(c,\mathbf{u})$	[m/s]	$\gamma = 1/\sqrt{[1 - (u/c)^2]}$, $\mathbf{u}$ = particle:group velocity
4-Momentum	$\mathbf{P} = P^\mu = (E/c,\mathbf{p})$	[kg·m/s = N·s]	E = relativistic energy, $\mathbf{p}$ = relativistic 3-momentum
4-WaveVector	$\mathbf{K} = K^\mu = (\omega/c,\mathbf{k} = \omega\hat{\mathbf{n}}/v_{phase})$	[{rad}/m]	$v_{phase} = \omega/k$ = phase velocity
4-TotalMomentum	$\mathbf{P_T} = P_T{}^\mu = (H/c=E_T/c,\mathbf{p_T})$	[kg·m/s]	H = Hamiltonian = E_T = Total Energy of System
4-TotalWaveVector	$\mathbf{K_T} = K_T{}^\mu = (\omega_T/c,\mathbf{k_T})$	[{rad}/m]	Waves are additive, as are Momenta (superposition)

Area of Closed Planar Curve in 3D space: $\mathbf{A} = (½)\oint(\mathbf{r} \times d\mathbf{r})$ [m^2] $\rightarrow A_z = (½)\oint (x\,dy - y\,dx)$ for {x-y plane}
3-angular-velocity: $\mathbf{\Omega} = (\mathbf{r} \times \mathbf{v})/r^2$ [1/s] 3-angular-momentum $\mathbf{L}=(\mathbf{r} \times \mathbf{p})=I\mathbf{\Omega} \sim mr^2\mathbf{\Omega}$
3-tangential-velocity (circular motion): $\mathbf{u}_{\mathbf{rotational}} = \mathbf{v}_\perp = \mathbf{\Omega} \times \mathbf{r}$ [m/s] velocity perpendicular (⊥) to center of rotation

$\mathbf{P} = (E/c,\mathbf{p}) = \hbar\mathbf{K} = \hbar(\omega/c,\mathbf{k} = \omega\hat{\mathbf{n}}/v_{phase})$ particle-wave duality [·⁝]

$\mathbf{P}\cdot d\mathbf{R} = \hbar\mathbf{K}\cdot d\mathbf{R}$ [kg·m²/s = J·s = Action] Action:Phase Relation General WaveFunction
$= dS = \hbar d\Phi$ [kg·m²/s = J·s = Action] $S_{action} = \hbar\Phi_{phase}$ $\Psi = \Psi_o e\hat{}(-i\Phi) = \Psi_o e\hat{}(-iS/\hbar)$

Lorentz Scalar Invariant Action S (additive subparts):
$S_{total} = S_{kinetic} + S_{EM} + S_{rotational} + S_{gravitational} + S_{other} + \ldots$

Lorentz Scalar Invariant Phase Φ (additive subparts):
$\Phi_{total} = \Phi_{kinetic} + \Phi_{EM} + \Phi_{rotational} + \Phi_{gravitational} + \Phi_{other} + \ldots$

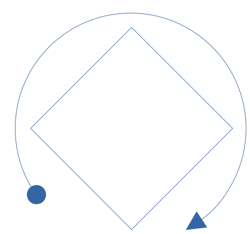
4-TotalMomentum = – 4-Gradient[Action] { $\mathbf{P_T} = -\partial[S_{action,total}]$ }
$-\partial[S_{total}] = -\partial[S_{kinetic}] + -\partial[S_{EM}] + -\partial[S_{rotational}] + -\partial[S_{gravitational}]$
$= \mathbf{P_T} = \mathbf{P} + q\mathbf{A_{EM}} + \mathbf{P_{rotational}} + \mathbf{P_{gravitational}}$

4-TotalWaveVector = – 4-Gradient[Phase] { $\mathbf{K_T} = -\partial[\Phi_{phase,total}]$ }
$-\partial[\Phi_{total}] = -\partial[\Phi_{kinetic}] + -\partial[\Phi_{EM}] + -\partial[\Phi_{rotational}] + -\partial[\Phi_{gravitational}]$
$= \mathbf{K_T} = \mathbf{K} + (q/\hbar)\mathbf{A_{EM}} + \mathbf{K_{rotational}} + \mathbf{K_{gravitational}}$

Action S (relativistic Invariant, Lorentz Scalar Product)
$S = -\int\mathbf{P}\cdot d\mathbf{R} = -\int\mathbf{P}\cdot(d\mathbf{R}/d\tau)d\tau = -\int(\mathbf{P}\cdot\mathbf{U})d\tau = -\int E_o d\tau = -m_o c^2\int d\tau$
$dS = -\mathbf{P}\cdot d\mathbf{R} = -(Edt - \mathbf{p}\cdot d\mathbf{r}) = (-Edt + \mathbf{p}\cdot d\mathbf{r})$
$dS_{temporal\ part} = -Edt$
$dS_{spatial\ part} = \mathbf{p}\cdot d\mathbf{r}$

Phase Φ (relativistic Invariant, Lorentz Scalar Product)
$\Phi = -\int\mathbf{K}\cdot d\mathbf{R} = -\int\mathbf{K}\cdot(d\mathbf{R}/d\tau)d\tau = -\int(\mathbf{K}\cdot\mathbf{U})d\tau = -\int\omega_o d\tau = -\omega_o\int d\tau$
$d\Phi = -\mathbf{K}\cdot d\mathbf{R} = -(\omega dt - \mathbf{k}\cdot d\mathbf{r}) = (-\omega dt + \mathbf{k}\cdot d\mathbf{r})$
$d\Phi_{temporal\ part} = -\omega dt$
$d\Phi_{spatial\ part} = \mathbf{k}\cdot d\mathbf{r}$

4-Momentum subpart $\mathbf{P_{rotational}}$ due to Rotation (not 4-AngularMomentum, which is $\boldsymbol{M} = \mathbf{R} \wedge \mathbf{P}$)
$\mathbf{P_{rotational}} = m_o\mathbf{U_{rotational}} = (E_{rotational}/c,\mathbf{p_{rotational}}) = m_o\gamma(c,\mathbf{u_{rotational}}) = m(c,\mathbf{u_{rotational}}) = (mc,m\mathbf{u_{rotational}})$



Take Spatial part:
$\mathbf{p_{rotational}}$
$= \gamma m_o\mathbf{u_{rotational}}$ $= m\mathbf{u_{rotational}}$ $\rightarrow m_o\mathbf{u_{rotational}}$ for { $|\mathbf{v}_\perp| \ll c$ }
$= \gamma m_o\mathbf{v}_\perp$ $= m\mathbf{v}_\perp$ $\rightarrow m_o\mathbf{v}_\perp$ for { $|\mathbf{v}_\perp| \ll c$ }
$= \gamma m_o(\mathbf{\Omega} \times \mathbf{r})$ $= m(\mathbf{\Omega} \times \mathbf{r})$ $\rightarrow m_o(\mathbf{\Omega} \times \mathbf{r})$ for { $|\mathbf{v}_\perp| \ll c$ }

$\Delta\Phi_{rotational}$ The rotational part of the Relativistic Phase
$= \oint d\Phi_{rotational}$ in differential Relativistic Phase form
$= (1/\hbar)\oint dS_{rotational}$ in differential Relativistic Action form { $S_{action} = \hbar\Phi_{phase}$ } from { $\mathbf{P} = \hbar\mathbf{K}$ }
$= (1/\hbar)\oint \mathbf{p_{rotational}}\cdot d\mathbf{r}$ in 3-momentum form
$= (1/\hbar)\oint m\mathbf{v}_\perp\cdot d\mathbf{r}$ in 3-tangential-velocity form
$= (1/\hbar)\oint m(\mathbf{\Omega} \times \mathbf{r})\cdot d\mathbf{r}$ in 3-angular-velocity form
$= (m/\hbar)\oint (\mathbf{\Omega} \times \mathbf{r})\cdot d\mathbf{r}$ mass unchanged by closed-path integral
$= (m/\hbar)\oint (\mathbf{r} \times d\mathbf{r})\cdot\mathbf{\Omega}$ by Vector Triple Product Rule
$= (m/\hbar)\mathbf{\Omega}\cdot\oint (\mathbf{r} \times d\mathbf{r})$ assume $\mathbf{\Omega}$ is constant
$= \underline{(2m/\hbar)\mathbf{\Omega}\cdot\mathbf{A}}$ Def. of Area of closed planar curve [kg]*[{rad}/(J·s)]*[1/s]*[m²] = {rad}
= Sagnac Effect (single-beam full-circle or split dual-beam half-circle, matter-wave form $\{m_o > 0\}$)
$= (2mc^2\omega/\hbar\omega c^2)\mathbf{\Omega}\cdot\mathbf{A}$
$= (2\omega/c^2)\mathbf{\Omega}\cdot\mathbf{A}$ { $mc^2 = \hbar\omega$ } particle-wave duality [·⁝]
$= (4\pi f/c^2)\mathbf{\Omega}\cdot\mathbf{A}$ { $\omega = 2\pi f$ } angular {radian:cycle} relation
$= \underline{(4\pi/\lambda c)\mathbf{\Omega}\cdot\mathbf{A}}$ { $\lambda f = c$ } {wavelength:frequency} relation for photons [{rad}/m]*[s/m]*[1/s]*[m²] = {rad}
= Sagnac Effect (single-beam full-circle or split dual-beam half-circle, photonic form $\{m_o = 0\}$)

= Often seen as $\underline{(4m/\hbar)\mathbf{\Omega}\cdot\mathbf{A}}$ or $\underline{(8\pi/\lambda c)\mathbf{\Omega}\cdot\mathbf{A}}$ when using split dual-beam full-circle, which gives (2×) the phase shift

SR derived Aharonov-Bohm:Aharonov-Casher Effect, Measuring Absolute EM Potentials, using 4D Tensors (esp. 4-Vectors):

4-Vector = 4D (1,0)-Tensor	**Properties**		
4-Position	$\mathbf{R} = R^\mu = (ct,\mathbf{r})$	[m]	c = Invariant LightSpeed
4-Differential	$d\mathbf{R} = dR^\mu = (cdt,d\mathbf{r})$	[m]	Calculus works normally on 4-Vectors
4-Velocity	$\mathbf{U} = U^\mu = \gamma(c,\mathbf{u})$	[m/s]	$\gamma = 1/\sqrt{[1 - (u/c)^2]}$, $\mathbf{u}$ = particle:group velocity
4-Momentum	$\mathbf{P} = P^\mu = (E/c,\mathbf{p})$	[kg·m/s = N·s]	E = relativistic energy, $\mathbf{p}$ = relativistic 3-momentum
4-WaveVector	$\mathbf{K} = K^\mu = (\omega/c,\mathbf{k} = \omega\hat{\mathbf{n}}/v_{phase})$	[{rad}/m]	$v_{phase} = \omega/k$ = phase velocity
4-TotalMomentum	$\mathbf{P_T} = P_T{}^\mu = (H/c=E_T/c,\mathbf{p_T})$	[kg·m/s]	H = Hamiltonian = Total Energy of System
4-TotalWaveVector	$\mathbf{K_T} = K_T{}^\mu = (\omega_T/c,\mathbf{k_T})$	[{rad}/m]	Waves are additive, as are Momenta (superposition)
4-VectorPotential	$\mathbf{A} = A^\mu = (\varphi/c,\mathbf{a})$	[kg·m/(C·s)]	φ = scalar-potential, $\mathbf{a}$ = 3-vector-potential
4-PotentialMomentum	$\mathbf{Q} = Q^\mu = (q\varphi/c,q\mathbf{a})$	[kg·m/s]	qφ = potential-energy, q$\mathbf{a}$ = 3-potential-momentum

$\mathbf{P} = (E/c,\mathbf{p}) = \hbar\mathbf{K} = \hbar(\omega/c,\mathbf{k} = \omega\hat{\mathbf{n}}/v_{phase})$ particle-wave duality [·⦚]

$\mathbf{P}\cdot d\mathbf{R} = \hbar\mathbf{K}\cdot d\mathbf{R}$ [kg·m²/s = J·s = Action]
$= dS = \hbar d\Phi$ [kg·m²/s = J·s = Action]

Action:Phase Relation
$S_{action} = \hbar\Phi_{phase}$

General WaveFunction
$\Psi = \Psi_o e\hat{}(-i\Phi) = \Psi_o e\hat{}(-iS/\hbar)$

Lorentz Scalar Invariant Action S (additive subparts):
$S_{total} = S_{kinetic} + S_{EM} + S_{rotational} + S_{gravitational} + S_{other} + \ldots$

4-TotalMomentum = – 4-Gradient[Action] { $\mathbf{P_T} = -\partial[S_{action,total}]$ }
$-\partial[S_{total}] = -\partial[S_{kinetic}] + -\partial[S_{EM}] + -\partial[S_{rotational}] + -\partial[S_{gravitational}]$
$= \mathbf{P_T} = \mathbf{P} + q\mathbf{A_{EM}} + \mathbf{P_{rotational}} + \mathbf{P_{gravitational}}$

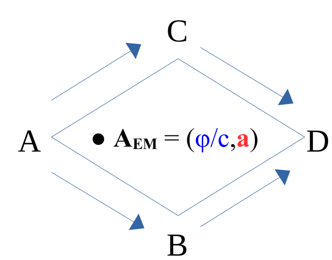
Action S (relativistic Invariant, Lorentz Scalar Product)
$S = -\int\mathbf{P}\cdot d\mathbf{R} = -\int\mathbf{P}\cdot(d\mathbf{R}/d\tau)d\tau = -\int(\mathbf{P}\cdot\mathbf{U})d\tau = -\int E_o d\tau = -m_o c^2\int d\tau$
$dS = -\mathbf{P}\cdot d\mathbf{R} = -(Edt - \mathbf{p}\cdot d\mathbf{r}) = (-Edt + \mathbf{p}\cdot d\mathbf{r})$
$dS_{temporal\ part} = -Edt$
$dS_{spatial\ part} = \mathbf{p}\cdot d\mathbf{r}$

Lorentz Scalar Invariant Phase Φ (additive subparts):
$\Phi_{total} = \Phi_{kinetic} + \Phi_{EM} + \Phi_{rotational} + \Phi_{gravitational} + \Phi_{other} + \ldots$

4-TotalWaveVector = – 4-Gradient[Phase] { $\mathbf{K_T} = -\partial[\Phi_{phase,total}]$ }
$-\partial[\Phi_{total}] = -\partial[\Phi_{kinetic}] + -\partial[\Phi_{EM}] + -\partial[\Phi_{rotational}] + -\partial[\Phi_{gravitational}]$
$= \mathbf{K_T} = \mathbf{K} + (q/\hbar)\mathbf{A_{EM}} + \mathbf{K_{rotational}} + \mathbf{K_{gravitational}}$

Phase Φ (relativistic Invariant, Lorentz Scalar Product)
$\Phi = -\int\mathbf{K}\cdot d\mathbf{R} = -\int\mathbf{K}\cdot(d\mathbf{R}/d\tau)d\tau = -\int(\mathbf{K}\cdot\mathbf{U})d\tau = -\int\omega_o d\tau = -\omega_o\int d\tau$
$d\Phi = -\mathbf{K}\cdot d\mathbf{R} = -(\omega dt - \mathbf{k}\cdot d\mathbf{r}) = (-\omega dt + \mathbf{k}\cdot d\mathbf{r})$
$d\Phi_{temporal\ part} = -\omega dt$
$d\Phi_{spatial\ part} = \mathbf{k}\cdot d\mathbf{r}$

For the AB Effect, examine 4-TotalMomentum:
$\mathbf{P_T} = \mathbf{P} + \mathbf{Q_{EM}} = \mathbf{P} + q\mathbf{A_{EM}}$ $\mathbf{P_T} = \hbar\mathbf{K_T}$ particle-wave duality [·⦚]
$\mathbf{P_T}/\hbar = \mathbf{P}/\hbar + \mathbf{Q_{EM}}/\hbar = \mathbf{P}/\hbar + q\mathbf{A_{EM}}/\hbar$
$\mathbf{K_T} = \mathbf{K} + \mathbf{Q_{EM}}/\hbar = \mathbf{K} + (q/\hbar)\mathbf{A_{EM}}$



$\mathbf{K_T} = \mathbf{K_{dyn}} + \mathbf{K_{A\text{-}B}}$

$d\Phi = -\mathbf{K}\cdot d\mathbf{R} = -(\omega dt - \mathbf{k}\cdot d\mathbf{r}) = (-\omega dt + \mathbf{k}\cdot d\mathbf{r})$

The Aharonov-Bohm Effect
$d\Phi_{AB} = -(q/\hbar)\mathbf{A_{EM}}\cdot d\mathbf{R} = -(q/\hbar)(\varphi dt - \mathbf{a_{EM}}\cdot d\mathbf{r}) = (q/\hbar)(\mathbf{a_{EM}}\cdot d\mathbf{r} - \varphi dt)$
Temporal (scalar) part: $d\Phi_{AB} = (q/\hbar)(-\varphi dt)$ {AB Electric Effect}
Spatial (3-vector) part: $d\Phi_{AB} = (q/\hbar)(\mathbf{a_{EM}}\cdot d\mathbf{r})$ {AB Magnetic Effect}

$\Delta\Phi_{AB} = -(q/\hbar)\oint\mathbf{A_{EM}}\cdot d\mathbf{R} = -(q/\hbar)\oint(\varphi dt - \mathbf{a_{EM}}\cdot d\mathbf{r}) = (q/\hbar)\oint(\mathbf{a_{EM}}\cdot d\mathbf{r} - \varphi dt)$
Temporal (scalar) part: $\Delta\Phi_{AB} = (q/\hbar)\oint(-\varphi dt)$ {AB Electric Effect, EM point charge q} [C{rad}/(J·s)*kg·m²/(C·s²)*s]={rad}
Spatial (3-vector) part: $\Delta\Phi_{AB} = (q/\hbar)\oint(\mathbf{a_{EM}}\cdot d\mathbf{r})$ {AB Magnetic Effect, EM point EM charge q} [C{rad}/(J·s)*kg·m/(C·s)*m]={rad}

The Aharonov-Casher Effect is similar, except using point-dipoles instead of point-monopoles (regular EM charges)
Mixed (3-vector) part: $\Delta\Phi_{AC} = (1/\hbar c^2)\oint([\mathbf{e} \times \boldsymbol{\mu}]\cdot d\mathbf{r})$ {AC Effect, EM magnetic-dipole $\boldsymbol{\mu}$} [{rad}/(J·s)*(s²/m²)*(N/C)*(J/T)*m]={rad}

Magnetic Flux $\Phi_B = \oint(\mathbf{a_{EM}}\cdot d\mathbf{r})$
$\Delta\Phi_{AB} = (q\Phi_B\,\hbar)$ {AB Magnetic Effect}

Gravitationally-induced neutron interference, Measuring Relative Gravity Potential, using 4D Tensors (esp. 4-Vectors):

4-Vector = 4D (1,0)-Tensor **Properties & Relations** — $(+,-,-,-)$ metric signature : $(cd\tau)^2 = g_{\mu\nu}dX^\mu dX^\nu$

4-Differential	$d\mathbf{X} = dX^\mu = (cdt,d\mathbf{x})$		[m]	
4-Velocity	$\mathbf{U} = U^\mu = \gamma(c,\mathbf{u})$	$= d\mathbf{X}/d\tau$	[m/s]	$\gamma = 1/\sqrt{[1 - \mathbf{u}\cdot\mathbf{u}/c^2]}$ in regular SR
4-WaveVector	$\mathbf{K} = K^\mu = (\omega/c,\mathbf{k})$	$= (\omega_o/c^2)\mathbf{U} = (1/\hbar)\mathbf{P}$	[{rad}/m]	
4-Momentum	$\mathbf{P} = P^\mu = (mc,\mathbf{p})$	$= m_o\mathbf{U} = \hbar\mathbf{K}$	[kg·m/s = N·s]	

GR "Weak-Field" limiting-case… or alternately viewed as a small perturbation field ($h_{\mu\nu}$) on the SR Minkowski Metric ($\eta_{\mu\nu}$):
SpaceTime Metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$: $\eta_{\mu\nu} = \mathrm{Diag}[1,-1,-1,-1]$: $h_{\mu\nu} = (2\varphi/c^2)\delta_{\mu\nu}$: $g_{00} = (1+2\varphi/c^2)$ & $g_{ii} = (-1+2\varphi/c^2)$: Gravity potential (φ)

$cd\tau = \sqrt{[g_{\mu\nu}dX^\mu dX^\nu]} = \sqrt{[g_{\mu\nu}(dX^\mu/dt)(dX^\nu/dt)]}dt = \sqrt{[(1+2\varphi/c^2)c^2+(-1+2\varphi/c^2)\mathbf{u}\cdot\mathbf{u}]}dt = \sqrt{[c^2+2\varphi-\mathbf{u}\cdot\mathbf{u}+2\varphi\mathbf{u}\cdot\mathbf{u}/c^2]}dt$

$d\tau = \sqrt{[1+2\varphi/c^2-\mathbf{u}\cdot\mathbf{u}/c^2+2\varphi\mathbf{u}\cdot\mathbf{u}/c^4]}dt \sim \sqrt{[1+2\varphi/c^2-\mathbf{u}\cdot\mathbf{u}/c^2+\{0...\}]}dt = dt/\gamma_{WeakGrav}$ Assume $O[1/c^4]$ factor ~ {0...}

So, "Weak-Gravity" Lorentz Factor $\gamma_{WeakGrav} = 1/\sqrt{[1+2\varphi/c^2-\mathbf{u}\cdot\mathbf{u}/c^2]}$ and the Metric effectively is: $g_{00} = (1+2\varphi/c^2)$ & $g_{ii} \to (-1)$
$\Delta\tau = \Delta t/\gamma_{WeakGrav} = \sqrt{[1+2\varphi/c^2-\mathbf{u}\cdot\mathbf{u}/c^2]}\Delta t$: Aging rate slows for: lower in a gravity potential ($\varphi = -\mathbf{g}\cdot\mathbf{z}$) and at higher velocity ($\mathbf{u}$)

$\mathbf{U}\cdot\mathbf{U} = U^\mu g_{\mu\nu}U^\nu = g_{\mu\nu}U^\mu U^\nu = (1+2\varphi/c^2)(u^0)^2 + (-1)(u^i\cdot u^j) = \gamma^2[(1+2\varphi/c^2)c^2 + (-1)\mathbf{u}\cdot\mathbf{u}] = \gamma^2[(c^2+2\varphi c^2/c^2)-(\mathbf{u}\cdot\mathbf{u})] = \gamma^2[c^2+2\varphi-\mathbf{u}\cdot\mathbf{u}]$
$= \gamma^2c^2[1+2\varphi/c^2-\mathbf{u}\cdot\mathbf{u}/c^2] = (c^2)$ if $\gamma = \gamma_{WeakGrav}$

4-WaveVector $\mathbf{K}$ Lorentz Invariant:
$\mathbf{K}\cdot\mathbf{K} = (\omega_o/c^2)^2\mathbf{U}\cdot\mathbf{U} = (\omega_o/c^2)^2\gamma^2c^2[1+2\varphi/c^2-\mathbf{u}\cdot\mathbf{u}/c^2] = (\omega/c)^2[1+2\varphi/c^2-\mathbf{u}\cdot\mathbf{u}/c^2] = [(\omega/c)^2+(\omega/c)^2 2\varphi/c^2-(\omega/c)^2\mathbf{u}\cdot\mathbf{u}/c^2] = (\omega_o/c)^2$

$\mathbf{K}\cdot\mathbf{K} = [(\omega/c)^2+(mc/\hbar)^2 2\varphi/c^2-(mc/\hbar)^2\mathbf{u}\cdot\mathbf{u}/c^2] = (\omega_o/c)^2$
$\mathbf{K}\cdot\mathbf{K} = [(\omega/c)^2+(m/\hbar)^2 2\varphi-(1/\hbar)^2\mathbf{p}\cdot\mathbf{p}] = (\omega_o/c)^2$
$\mathbf{K}\cdot\mathbf{K} = [(\omega/c)^2+(m/\hbar)^2 2\varphi-\mathbf{k}\cdot\mathbf{k}] = (\omega_o/c)^2$

$\mathbf{P} = (mc=E/c,\mathbf{p}) = \hbar\mathbf{K} = \hbar(\omega/c,\mathbf{k})$ particle-wave duality [·⦚]
Temporal part: $[\omega/c = mc/\hbar] \to [(\omega/c)^2 = (mc/\hbar)^2]$
Spatial part: $[\mathbf{p} = \hbar\mathbf{k}]$

$[(\omega/c)^2+(m/\hbar)^2 2\varphi-\mathbf{k}\cdot\mathbf{k}]_{Path\ AB} = [(\omega'/c)^2+(m/\hbar)^2 2\varphi'-\mathbf{k'}\cdot\mathbf{k'}]_{Path\ CD}$ because $\mathbf{K}\cdot\mathbf{K} = \mathbf{K'}\cdot\mathbf{K'}$: The gravitational difference is accounted for.

Assume that angular frequency (ω), and hence energy ($E = \hbar\omega$) of particle-wave, remains unchanged ($\omega = \omega'$) or ($E = \hbar\omega = \hbar\omega' = E'$).
This is a condition of neutrons having essentially elastic collisions with the reflectors; just change direction, not magnitude.

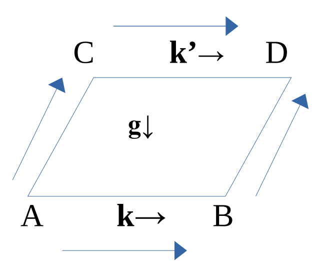
$[(m/\hbar)^2 2\varphi-\mathbf{k}\cdot\mathbf{k}]_{Path\ AB} = [(m/\hbar)^2 2\varphi'-\mathbf{k'}\cdot\mathbf{k'}]_{Path\ CD}$

$k'^2 = k^2 - (m/\hbar)^2 2\Delta\varphi$

$k' = k\sqrt{[1 - (m/\hbar k)^2 2\Delta\varphi]} \sim k[1-(m/\hbar k)^2\Delta\varphi]$: Assumes small mass m, small potential change $\Delta\varphi$, large wave number k

$k' - k = k[1-(m/\hbar k)^2\Delta\varphi)] - k = k-(m/\hbar)^2\Delta\varphi/k - k = -(m/\hbar)^2\Delta\varphi/k = -(m/\hbar)^2\Delta\varphi\bar{\lambda} = -(m/\hbar)^2\Delta\varphi\lambda/2\pi$

using wavenumber $k = 1/\bar{\lambda} = 2\pi/\lambda$



Compare Paths ABD vs. ACD: Source at A and Detector at D
Phase Difference $\Delta\Phi = \oint\mathbf{K}_{Path}\cdot d\mathbf{R} = \int\mathbf{K}_{AB}\cdot d\mathbf{R}_{AB} + \int\mathbf{K}_{BD}\cdot d\mathbf{R}_{BD} - \int\mathbf{K}_{AC}\cdot d\mathbf{R}_{AC} - \int\mathbf{K}_{CD}\cdot d\mathbf{R}_{CD}$

Paths AC & BD cancel each other since $\mathbf{K}_{AC}=\mathbf{K}_{BD}$, Paths AB and CD both have length (L).

$\Delta\Phi = \int\mathbf{K}_{AB}\cdot d\mathbf{R}_{AB} - \int\mathbf{K}_{CD}\cdot d\mathbf{R}_{CD} = (\omega_{AB}T-\mathbf{k}_{AB}L) - (\omega_{CD}T-\mathbf{k}_{CD}L) = (\mathbf{k}_{CD} - \mathbf{k}_{AB})L$, as temporal components cancel.

$\Delta\Phi = (\mathbf{k}_{CD} - \mathbf{k}_{AB})L = -(m/\hbar)^2\Delta\varphi\lambda L/2\pi = -(m/\hbar)^2 g\Delta z\lambda L/2\pi = -(m/\hbar)^2 gH\sin(\alpha)\lambda L/2\pi = -(m/\hbar)^2 gA_o\sin(\alpha)\lambda/2\pi$

using: Gravitational Potential Difference $\Delta\varphi=g*\Delta z$, Height Difference Δz = Tiltable Height $H*\sin(\alpha)$, Path Area $A_o=L*H$

Phase Difference: start at A, end at D: $\Delta\Phi = -(m/\hbar)^2 gA_o\sin(\alpha)\lambda/2\pi$

This matches the COW (R. Collela, A.W. Overhauser, S.A. Werner) Gravitationally-Induced Quantum Interference result.
This is also a micro-scale test of relativity's Equivalence Principle.
Note: Binomial Approximation $(1+x)^n \sim (1+nx)$ for $|nx| \ll 1$
Note: By Green's Theorem, I believe the shape of the simple-closed-path-area is independent of the result.

To recap, the SR 4-Vectors have some very simple, fundamental, and Invariant Lorentz Scalar relations between one another:

4-Position $\mathbf{R} = R^{\mu} = (ct,\mathbf{r}) \in$ <Event> $\in$ <Time·Space> $\rightarrow (E_o/c^2) \twoheadrightarrow \mathbf{P} = (E/c,\mathbf{p})$

4-Velocity $\mathbf{U} = U^{\mu} = \gamma(c,\mathbf{u}) = (\mathbf{U}\cdot\partial)\mathbf{R} = (d/d\tau)\mathbf{R}$ /

4-Momentum $\mathbf{P} = P^{\mu} = (E/c,\mathbf{p}) = (m_o)\mathbf{U}$ $\mathbf{U} = \gamma(c,\mathbf{u})$ $(\hbar)\uparrow \downarrow(1/\hbar)$

4-WaveVector $\mathbf{K} = K^{\mu} = (\omega/c,\mathbf{k}) = (1/\hbar)\mathbf{P}$ \

4-Gradient $\partial = \partial^{\mu} = (\partial_t/c,-\nabla) = (-i)\mathbf{K}$ $\rightarrow (\omega_o/c^2) \rightsquigarrow \mathbf{K} = (\omega/c,\mathbf{k})$

The main point of work above is to show that the last two relations, $\{\mathbf{K} = (1/\hbar)\mathbf{P}\}$ and $\{\partial = -i\mathbf{K}\}$, alternately $\{\mathbf{P} = \hbar\mathbf{K}\}$ and $\{\mathbf{K} = i\partial\}$, are of the same character as the other relations in this group. They are derivable from SR and/or purely-mathematical principles, and do not require quantum axioms for their existence. These will then lead to wave-particle duality and other derived quantum "axioms".

The Lorentz Scalar Products of these SR-related 4-Vectors gives the following chain of Invariants:

$(\mathbf{R}\cdot\mathbf{R}) = (c\tau)^2$

$(\mathbf{U}\cdot\mathbf{U}) = (c)^2$

$(\mathbf{P}\cdot\mathbf{P}) = (m_oc)^2$

$(\mathbf{K}\cdot\mathbf{K}) = (m_oc/\hbar)^2 = (\omega_o/c)^2 = (1/\lambdabar)^2$: $(\hbar/m_oc) = (c/\omega_o) = \lambdabar = \lambda/2\pi$ is the reduced Compton wavelength [length = m]

$(\partial\cdot\partial) = (im_oc/\hbar)^2 = -(m_oc/\hbar)^2 = -(\omega_o/c)^2 = -(1/\lambdabar)^2$: The fundamental quantum Klein-Gordon (KG) RQM wave relation: **[SR→QM]**

Each step is a logical progression, taking into account the simple relation between each of these SR 4-Vectors.
In the same way that the Relativistic 4D Euler-Lagrange Relation $(\mathbf{U}\cdot\partial_{\mathbf{R}})[\partial_{\mathbf{U}}] = (d/d\tau)[\partial_{\mathbf{U}}] = \partial_{\mathbf{R}}$ {itself a variation of $(d/d\tau)[\mathbf{R}] = \mathbf{U}$} implies that there can exist a Lagrangian function (L) that solves it, the KG relation $(\partial\cdot\partial) = (im_oc/\hbar)^2 = -(m_oc/\hbar)^2 = -(\omega_o/c)^2$ implies that there can exist a "wavefunction" (Ψ) which solves it. One does not need a presupposed quantum axiom. The Klein-Gordon relation gives a relativistic, 2nd order, linear Partial Differential Equation (PDE). The fact that it is a linear PDE leads to the principle of quantum superposition. The standard Schrödinger quantum wave equation $\{ (i\hbar\partial_t) = [(V) - (\hbar\nabla)^2/(2m_o)] \}$ is the non-relativistic ($|\mathbf{v}| \ll c$) limit-case of the KG relativistic quantum wave equation, which continues to show superposition.

The Klein-Gordon Eqn. $\{(\partial\cdot\partial) - (im_oc/\hbar)^2 = 0\}$ is itself the Relativistic Quantum (RQM) Equation for spin=0 particles (4-Scalars).
Factoring the KG Eqn. $\{(\partial + im_oc/\hbar)\cdot(\partial - im_oc/\hbar) = 0\}$ leads to the RQM Dirac Eqn. for spin=1/2 particles (4-Spinors).
Applying the KG Eqn. to a 4-Vector field leads to the RQM Proca Eqn. for spin=1 particles (4-Vectors).
Taking the low-velocity-limit ($|\mathbf{v}| \ll c$) of the KG leads to the standard QM non-relativistic Schrödinger Eqn., for spin=0 (4-Scalar).
Taking the low-velocity-limit ($|\mathbf{v}| \ll c$) of the Dirac leads to the standard QM non-relativistic Pauli Eqn., for spin=1/2 (4-Spinor).
Setting RestMass $\{m_o \rightarrow 0\}$ gives the RQM Free Wave (4-Scalar), Weyl (4-Spinor), and Free Maxwell Standard EM (4-Vector) Eqns.
In all of these cases, the equations can be modified to work with various potentials/interactions by using more SR 4-Vectors, and more empirically-found relations between them, ex. the Minimal Coupling Relations $\{\mathbf{P} = \mathbf{P_T} - q\mathbf{A}\}$: Here $\mathbf{P_T}$ means (dynamic + EM field).
4-TotalMomentum $\{\mathbf{P_T} = (H/c{=}E_T/c,\mathbf{p_T}) = \mathbf{P} + q\mathbf{A} = ([E+q\varphi]/c,\mathbf{p}+q\mathbf{a})\}$, with 4-Momentum $\mathbf{P}$, EM charge (q), & 4-VectorPotential $\mathbf{A}$
Also note that generating QM from RQM (via a low-energy limit) is much more natural and mathematically well-defined than attempting to "relativize" or "generalize" a given NRQM equation. Facts assumed from a non-relativistic equation may or may not be applicable to a relativistic one, whereas relativistic facts are still true (though perhaps not apparent) in the low-velocity limiting-cases. This leads again to the idea that QM is an approximation-only of the more general RQM, just as SR is an approximation-only of GR.

The standard Schrödinger QM Relations derived from SR:

Again, we examine these SR 4-Vector relations derived above… and by simply combining them...
$\mathbf{P} = \hbar\mathbf{K}$ (a relation which is entirely empirical, based on just SR arguments, shown above)
$\mathbf{K} = i\partial$ (which is a relation for complex plane-waves, used in classical EM and elsewhere)

$\mathbf{P} = i\hbar\partial = (E/c,\mathbf{p}) = i\hbar(\partial_t/c,-\nabla)$
The temporal part $\{E = i\hbar\partial_t = i\hbar\, \partial/\partial t\}$ gives unitary QM time evolution operator. Again, emphasis: derived from SR.
The spatial part $\{\mathbf{p} = -i\hbar\nabla\}$ gives the QM momentum operator. Again, emphasis: derived from SR.

These are the main quantum relations used in standard QM calculations, as well as in RQM.
Just as a note, to emphasize again the SR origin of the 4-Vectors:
The 4-Momentum $\mathbf{P}$ is used in purely-relativistic particle collision calculations.
The 4-WaveVector $\mathbf{K}$ is used in purely-relativistic Doppler effect calculations, and in classical EM wave descriptions.
Both $\mathbf{P}$ and $\mathbf{K}$ are used in the relativistic Compton effect photon-electron scattering calculations.
The 4-Gradient ∂ is used in several purely-relativistic settings: charge conservation $(\partial\cdot\mathbf{J}) = 0$, particle # conservation $(\partial\cdot\mathbf{N}) = 0$
Lorenz EM Gauge $(\partial\cdot\mathbf{A}) = 0$, invariant d'Alembertian $(\partial\cdot\partial)$, proper time derivative $(\mathbf{U}\cdot\partial)$, Euler-Lagrange Equation $(d/d\tau)[\partial_{\mathbf{U}}] = \partial_{\mathbf{R}}$
Minkowski Metric $\partial^{\mu}[R^{\nu}] = \eta^{\mu\nu}$, SR SpaceTime Dimension $(\partial\cdot\mathbf{R}) = 4$, Lorentz-Transform $\partial_{\nu}[R^{\mu'}] = \partial R^{\mu'}/\partial R^{\nu} = \Lambda^{\mu'}{}_{\nu}$ etc.
These facts show that the tensorial 4-Vectors and their relations are from SR, without the need of QM axioms.

Non-zero SR→QM Commutation Relation between 3-position $\mathbf{x} = x^j$ and 3-momentum $\mathbf{p} = p^k$:

4-Position $\mathbf{R} = R^\mu = (ct,\mathbf{r}) = (ct,x,y,z) = \mathbf{X} = X^\mu$ — $\mathbf{X}$ is common alt. form of $\mathbf{R}$

4-Gradient $\partial = \partial^\mu = (\partial_t/c,-\nabla) = (\partial/_{c\partial t},-\partial/_{\partial x},-\partial/_{\partial y},-\partial/_{\partial z}) = (\partial_t/c,-\partial_x,-\partial_y,-\partial_z) = \partial/\partial X_\mu$ — with $X_\mu = \eta_{\mu\nu}X^\nu$

Let { f } be an arbitrary SR function.

$\mathbf{X}[f] = \mathbf{X}f$ & $\partial[f] = \partial[f]$ are the primitive relations. The following is basic calculus:

$\mathbf{X}[\partial[f]] = \mathbf{X}\partial[f]$	: Apply 1st primitive rule rightward, with 2nd primitive rule as the argument
$\partial[\mathbf{X}[f]] = \partial[\mathbf{X}f] = \partial[\mathbf{X}]f + \mathbf{X}\partial[f]$	: Calculus Product Rule, d[a*b] = d[a]*b+a*d[b], applies to differential & partial
$\partial[\mathbf{X}f] - \mathbf{X}\partial[f] = \partial[\mathbf{X}]f$	: Rearrange equation
$\partial[\mathbf{X}[f]] - \mathbf{X}[\partial[f]] = \partial[\mathbf{X}]f$	: Apply 1st primitive rule leftward to form a commutation relation

The 4-Vector parts of the lefthand side { $\partial[\mathbf{X}[f]] - \mathbf{X}[\partial[f]]$ } form a commutator { $[\,\mathbf{A}\,,\,\mathbf{B}\,]f = \mathbf{A}[\mathbf{B}[f]] - \mathbf{B}[\mathbf{A}[f]]$ }:
$\partial[\mathbf{X}[f]] - \mathbf{X}[\partial[f]] = [\,\partial\,,\,\mathbf{X}\,]f$

The 4-Vector parts of the righthand side { $\partial[\mathbf{X}]$ } can be computed, which leads to a tensor product with 2 indices (a dyadic):
$\partial[\mathbf{X}] = \partial^\mu[X^\nu] = (\partial_t/c,-\nabla)[(ct,\mathbf{r})] \rightarrow (\partial_t/c,-\partial_x,-\partial_y,-\partial_z)[(ct,x,y,z)]_{(Cartesian)} = \mathrm{Diag}[+1,-1,-1,-1]_{(Cartesian)} = \eta^{\mu\nu}$ = Minkowski Metric

And since { f } was an arbitrary SR function, we can remove it (or set it to unity), which leaves the functional form:
$[\,\partial\,,\,\mathbf{X}\,] = [\,\partial^\mu\,,\,X^\nu\,] = \partial^\mu[X^\nu] = \eta^{\mu\nu}$ = Minkowski Metric [dimensionless units]

At this point, we have established purely mathematically, that there exists in SR a non-zero commutation relation between the SR 4-Gradient ∂ and SR 4-Position $\mathbf{X}$.
Note also that { $\mathbf{X}[f] = \mathbf{X}f$ } does not actually say that $\mathbf{X}$ is necessarily an operator. It just says that {an $\mathbf{X}$ next to an f} = {$\mathbf{X}$ times f}.
The 4-Position $\mathbf{X}$ could be an operator or just a numerical vector.
The 4-Gradient ∂ is definitely an operator, because it is already an operator:function in pure SR, and uses basic calculus rules.

Now, using these 4-Vectors and the relations between them derived from SR earlier in this document:
4-Position $\mathbf{R} = R^\mu = (ct,\mathbf{r}) = (x^0,x^i) = \mathbf{X} = X^\mu$
4-Momentum $\mathbf{P} = P^\mu = (E/c,\mathbf{p}) = (p^0,p^i) = \hbar\mathbf{K}$ (a relation which is entirely empirical, based on just SR arguments, shown above)
4-WaveVector $\mathbf{K} = K^\mu = (\omega/c,\mathbf{k}) = (k^0,k^i) = i\partial$ (which is a relation for complex plane-waves, used in classical EM)
4-Gradient $\partial = \partial^\mu = (\partial_t/c,-\nabla) = (\partial^0,\partial^i) = \partial/\partial X_\mu$ with $X_\mu = \eta_{\mu\nu}X^\nu$

$[\partial,\mathbf{X}] = [\partial^\mu,X^\nu] = \eta^{\mu\nu}$
$[i\partial,\mathbf{X}] = [i\partial^\mu,X^\nu] = i\eta^{\mu\nu}$
$[\mathbf{K},\mathbf{X}] = [K^\mu,X^\nu] = i\eta^{\mu\nu}$ Using $\mathbf{K} = i\partial$
$[\hbar\mathbf{K},\mathbf{X}] = [\hbar K^\mu,X^\nu] = i\hbar\eta^{\mu\nu}$
$[\mathbf{P},\mathbf{X}] = [P^\mu,X^\nu] = i\hbar\eta^{\mu\nu}$ Using $\mathbf{P} = \hbar\mathbf{K}$
$[\mathbf{X},\mathbf{P}] = [X^\mu,P^\nu] = -i\hbar\eta^{\mu\nu}$ This is a major result of **[SR→QM]**.
The temporal part $[x^0,p^0] = [ct,E/c] = [t,E] = -i\hbar\eta^{00} = -i\hbar$ is the "oft-misunderstood" time-energy commutation relation.
The spatial part $[x^j,p^k] = i\hbar\delta^{jk}$ is the standard QM Canonical Commutation Relation, derived, not an axiom.
The mixed parts $[x^0,p^k] = [x^j,p^0] = \eta^{0k} = \eta^{j0} = 0$, meaning these parts commute normally, as expected classically.

Similar 4-Vector arguments lead to the standard angular-momentum quantum commutation relations via SR 4-AngularMomentum $M^{\mu\nu} = X^\mu \wedge P^\nu = X^\mu P^\nu - P^\nu X^\mu$. In fact, the entire Poincaré Algebra (Lie Algebra of the Poincaré Group) can be generated in this fashion.
P^μ is generator of SpaceTime-Translations (ΔX^μ). $M^{\mu\nu}$ is the generator of Lorentz-Transformations ($\Lambda^\mu{}_\nu$). $\eta^{\mu\nu}$ is the Minkowski Metric.

Canonical (Momentum, Position):	$[P^\mu,X^\nu] = i\hbar\eta^{\mu\nu}$	from $[\partial^\mu,X^\nu] = \eta^{\mu\nu}$
[Linear, Linear] Momentum:	$[P^\mu,P^\nu] = 0^{\mu\nu}$	from partials commuting $[\partial^\mu,\partial^\nu] = 0^{\mu\nu}$
[Angular, Linear] Momentum:	$[M^{\mu\nu},P^\sigma] = i\hbar(\eta^{\sigma\nu}P^\mu - \eta^{\sigma\mu}P^\nu)$	from $O^{\mu\nu} = X^\mu \wedge \partial^\nu$: $[O^{\mu\nu},P^\sigma] = (\eta^{\sigma\nu}P^\mu - \eta^{\sigma\mu}P^\nu)$
[Angular, Angular] Momentum:	$[M^{\mu\nu},M^{\sigma\rho}] = i\hbar(\eta^{\nu\sigma}M^{\mu\rho} + \eta^{\sigma\mu}M^{\rho\nu} + \eta^{\mu\rho}M^{\nu\sigma} + \eta^{\rho\nu}M^{\sigma\mu})$	also $[X^\mu,X^\nu] = 0^{\mu\nu}$ because just numbers

Again, the (i) and ($\hbar$) come from SR. The algebra is all real and overall dimensionless when using only {$\mathbf{X}$ and ∂} in the definitions.
Likewise, the general mathematical uncertainty relations, { $\sigma_A^2\sigma_B^2 \geq (1/2)|\langle[\mathbf{A},\mathbf{B}]\rangle|$ }, based on commutation relations, lead to the standard physical quantum Heisenberg uncertainty relations. Also note that the commutator order of operations is in accord with SR causality conditions. While spacelike-separated events |here⟩ and |there⟩ may occur in any temporal order, all observers will see the same temporal order of timelike-separated events |now⟩ and |then⟩, with |past-then⟩able to causally affect |now⟩ and |now⟩able to causally affect |future-then⟩. Thus, for time-like separations, if measurement-event **A** occurs temporally before measurement-event **B**, then this would be written in operator notation as: $|\Psi'\rangle = \mathbf{A}|\Psi\rangle$ then $|\Psi''\rangle = \mathbf{B}|\Psi'\rangle = \mathbf{BA}|\Psi\rangle$. The operator order shows the timelike-separated order of measurement-events. Due to non-zero commutation relations, $\mathbf{AB}|\Psi\rangle$ would give a different result.

Using Green's Vector Identity to establish a Conserved Current (could be # {dust or probability}or charged):
Consider the following purely mathematical argument:
$\partial\cdot(f\, \partial[g] - \partial[f]\, g) = f\, \partial\cdot\partial[g] - \partial\cdot\partial[f]\, g$ with (f) and (g) as SR functions

Proof of the 4-Divergence relation:
=========================
$\partial\cdot(f\, \partial[g] - \partial[f]\, g)$
$= \partial\cdot(f\, \partial[g]) - \partial\cdot(\partial[f]\, g)$
$= (f\, \partial\cdot\partial[g] + \partial[f]\cdot\partial[g]) - (\partial[f]\cdot\partial[g] + \partial\cdot\partial[f]\, g)$
$= f\, \partial\cdot\partial[g] - \partial\cdot\partial[f]\, g$
We can also multiply this by a constant Lorentz Invariant Scalar Constant (s), for dimensional-unit purposes.
$s\,(f\, \partial\cdot\partial[g] - \partial\cdot\partial[f]\, g) = s\, \partial\cdot(f\, \partial[g] - \partial[f]\, g) = \partial\cdot[s(f\, \partial[g] - \partial[f]\, g)] = \partial\cdot\mathbf{J}$
=========================
Thus there mathematically exists a 4-Current(Density) **J** derivable from the SR d'Alembertian ($\partial\cdot\partial$)

Now, applied to SR physics... Start with the Klein-Gordon relation derived above from the Lorentz Scalar Product:
$\partial\cdot\partial = (-im_oc/\hbar)^2 = -(m_oc/\hbar)^2$
$\partial\cdot\partial + (m_oc/\hbar)^2 = 0$

Let it act on SR function g
$\partial\cdot\partial[g] + (m_oc/\hbar)^2[g] = 0\,[g]$
Then pre-multiply by f
$[f]\partial\cdot\partial[g] + [f]\,(m_oc/\hbar)^2[g] = [f]\,0\,[g]$
$[f]\partial\cdot\partial[g] + (m_oc/\hbar)^2[f][g] = 0$

Let it act on SR function f
$\partial\cdot\partial[f] + (m_oc/\hbar)^2[f] = 0\,[f]$
Then post-multiply by g
$\partial\cdot\partial[f][g] + (m_oc/\hbar)^2[f][g] = 0\,[f][g]$
$\partial\cdot\partial[f][g] + (m_oc/\hbar)^2[f][g] = 0$

Now, subtract the two equations
$\{[f]\, \partial\cdot\partial[g] + (m_oc/\hbar)^2[f][g] = 0\} - \{ \partial\cdot\partial[f][g] + (m_oc/\hbar)^2[f][g] = 0\}$
$[f]\, \partial\cdot\partial[g] + (m_oc/\hbar)^2[f][g] - \partial\cdot\partial[f][g] - (m_oc/\hbar)^2[f][g] = 0$
$f\, \partial\cdot\partial[g] - \partial\cdot\partial[f]\, g = 0$

As noted from the mathematical Green's Vector Identity, this can be written as a 4-Divergence
with the additional constraint that it now also equates to 0, meaning that it is a conserved 4-Current **J**.
$s\, \partial\cdot(f\, \partial[g] - \partial[f]\, g) = \partial\cdot[s(f\, \partial[g] - \partial[f]\, g)] = \partial\cdot\mathbf{J} = 0$ This may be viewed as a generalized conserved flux.

Thus, there exists a conserved current-type 4-Vector, $\mathbf{J}_{\mathbf{prob}} = (\rho_{prob}c,\mathbf{j}_{\mathbf{prob}}) = s(f\, \partial[g] - \partial[f]\, g)$, for which $\partial\cdot\mathbf{J}_{\mathbf{prob}} = (\partial_t\, \rho_{prob} + \nabla\cdot\mathbf{j}_{\mathbf{prob}}) = 0$,
and which also solves the Klein-Gordon relation $(\partial\cdot\partial) = -(m_oc/\hbar)^2$, and gives unitary evolution and conservation of probability.

For generality, choose the SR 4-Vector relation ($\partial = -i\mathbf{K}$) as before with a complex planewave function $g = (a)e\text{^}{-i(\mathbf{K}\cdot\mathbf{X})} = \psi$,
and choose $f = g^* = (a)e\text{^}{i(\mathbf{K}\cdot\mathbf{X})} = \psi^*$ as its complex conjugate.
At this point, we can choose $s = (i\hbar/2m_o) = (ic^2/2\omega_o)$, which is Lorentz Scalar Invariant, in order to make the probability have [dimensionless units = #] and be normalized to unity in the rest case. In 4-Vector form this gives probability-density ρ_{prob} [#/m³].
$\mathbf{J}_{\mathbf{prob}} = (\rho_{prob}c,\mathbf{j}_{\mathbf{prob}}) = (i\hbar/2m_o)(\psi^*\partial[\psi] - \partial[\psi^*]\psi) = (ic^2/2\omega_o)(\psi^*\partial[\psi] - \partial[\psi^*]\psi)$ [#-flux = #/(m²·s) = (#/m³)·(m/s)]
$(\partial\cdot\mathbf{J}_{\mathbf{prob}}) = (s)\partial\cdot(\psi^*\partial[\psi] - \partial[\psi^*]\psi) = (s)(\partial[\psi^*]\cdot\partial[\psi] + \psi^*\partial\cdot\partial[\psi] - \partial\cdot\partial[\psi^*]\psi - \partial[\psi^*]\cdot\partial[\psi]) = (s)(\psi^*\partial\cdot\partial[\psi] - \partial\cdot\partial[\psi^*]\psi) = 0$

Examine the temporal component, the Relativistic Probability Density
$\rho_{prob}c = (i\hbar/2m_o)(\psi^*\, (\partial_t/c)[\psi] - (\partial_t/c)[\psi^*]\, \psi) = (ic^2/2\omega_o)(\psi^*\, (\partial_t/c)[\psi] - (\partial_t/c)[\psi^*]\, \psi)$
$\rho_{prob} = (i\hbar/2m_oc^2)(\psi^*\, \partial_t[\psi] - \partial_t[\psi^*]\, \psi) = (i/2\omega_o)(\psi^*\, \partial_t[\psi] - \partial_t[\psi^*]\, \psi)$
Assume wave solution in following general form:
$\{ \psi = A\, f[k]\, e^{(-i\omega t)} \}$ & $\{ \psi^* = A^*\, f[k]^*\, e^{(+i\omega t)} \}$
$\{ \partial_t[\psi] = (-i\omega)\, A\, f[k]\, e^{(-i\omega t)} = (-i\omega)\psi \}$ & $\{ \partial_t[\psi^*] = (+i\omega)\, A^*\, f[k]^*\, e^{(+i\omega t)} = (+i\omega)\psi^* \}$
then
$\rho_{prob} = (i/2\omega_o)(\psi^*\, \partial_t[\psi] - \partial_t[\psi^*]\, \psi)$
$\rho_{prob} = (i/2\omega_o)((-i\omega)\psi^*\, \psi - (+i\omega)\psi^*\, \psi)$
$\rho_{prob} = (i/2\omega_o)((-2i\omega)\psi^*\, \psi)$
$\rho_{prob} = (\omega/\omega_o)(\psi^*\, \psi)$
$\rho_{prob} = (\gamma\omega_o/\omega_o)(\psi^*\, \psi)$
$\rho_{prob} = (\gamma)(\psi^*\, \psi) = (\gamma)(\rho_{prob\, o})$
Finally, multiply by charge (q) to get standard SR EM 4-CurrentDensity = 4-ChargeFlux = $\mathbf{J} = (\rho c,\mathbf{j}) = q\mathbf{J}_{\mathbf{prob}} = q(\rho_{prob}c,\mathbf{j}_{\mathbf{prob}})$
One can generalize (in this case) to include the effects of an EM VectorPotential $\mathbf{A} = (\varphi/c,\mathbf{a})$
4-ProbabilityCurrentDensity $\mathbf{J}_{\mathbf{prob+EM}} = (\rho_{prob+EM}c,\ \mathbf{j}_{\mathbf{prob+EM}}) = (i\hbar/2m_o)(\psi^*\partial[\psi]-\partial[\psi^*]\psi) + (q/m_o)(\psi^*\, \psi)\mathbf{A}$

Examine the temporal component:
$\rho_{prob} = (i\hbar/2m_o c^2)(\psi^* \partial_t[\psi] - \partial_t[\psi^*]\psi) + (q/m_o)(\psi^*\psi)(\varphi/c^2)$
$\rho_{prob} \to (\gamma)(\psi^*\psi) + (\gamma)(q\varphi_o/m_o c^2)(\psi^*\psi) = (\gamma)[1 + q\varphi_o/E_o](\psi^*\psi)$
Typically, particle EM potential energy ($q\varphi_o$) is much less than particle rest energy (E_o), else it could generate new particles.
So, take ($q\varphi_o \ll E_o$), which gives the EM factor $(q\varphi_o/E_o) \sim 0$
Now, taking the low-velocity limit ($\gamma \to 1$), $\rho_{prob} = \gamma[1 + \sim 0](\psi^*\psi)$, $\rho_{prob} \to (\psi^*\psi) = (\rho_{probo})$ for { $|\mathbf{v}| \ll c$ }
The Standard Born Probability Interpretation, $(\psi^*\psi) = (\rho_{prob})$, only applies in the low-potential-energy & low-velocity limit
This is why the {non-positive-definite} probabilities and {|probabilities| > 1} in the RQM Klein-Gordon equation puzzled physicists, and is the reason why one must regard the probabilities as charge conservation instead.
The original definition from SR is Continuity of Worldlines, $(\partial \cdot \mathbf{J}_{prob}) = 0$, for which all is good and well in the RQM version.
The definition says there are no external sources or sinks of probability = conservation of probability.
The Born idea that Total $(\rho_{prob}) \to \mathrm{Sum}[(\psi^*\psi)] = 1$ is just the low-velocity ($|\mathbf{v}| \ll c$) QM limit-case.
Only the non-EM, classical, rest-version has Total $(\rho_{prob\ o}) = \mathrm{Sum}[(\psi^*\psi)] = 1$.
It is not a fundamental axiom, it is an emergent property which is valid only in the NRQM limit.
We now multiply by EM charge (q) to get:
4-"Charge"CurrentDensity $\mathbf{J} = (\rho c, \mathbf{j}) = q\mathbf{J}_{prob} = q(\rho_{prob}c, \mathbf{j}_{prob})$, which is the standard SR EM 4-CurrentDensity

Comparison of SR 4-(Dust)NumberFlux **N** to QM 4-ProbabilityCurrent $\mathbf{J}_{prob}$: the **same** 4-Vector:
SR 4-Vector (properties & relations):
4-Velocity $\mathbf{U} = \gamma(c, \mathbf{u})$ [m/s]
4-Gradient $\partial = \partial^\mu = (\partial_t/c, -\nabla) = (\partial/\partial R_\mu)$ [1/m]
4-(Dust)NumberFlux $\mathbf{N} = (nc, \mathbf{n}=n\mathbf{u}) = n(c, \mathbf{u}) = n_o\gamma(c, \mathbf{u}) = n_o\mathbf{U} = \mathbf{J}/q$ $[\#/(m^2 \cdot s) = (\#/m^3)\cdot(m/s) = \#\text{-flux}]$
4-Current(Density)=4-ChargeFlux $\mathbf{J} = (\rho c, \mathbf{j}=\rho\mathbf{u}) = \rho(c, \mathbf{u}) = \rho_o\gamma(c, \mathbf{u}) = qn_o\gamma(c, \mathbf{u}) = \rho_o\mathbf{U} = qn_o\mathbf{U} = q\mathbf{N}$ $[C/(m^2 \cdot s) = (C/m^3)\cdot(m/s) = q\text{-flux}]$
4-ProbabilityCurrentDensity $\mathbf{J}_{prob} = (\rho_{prob}c, \mathbf{j}_{prob}=\rho_{prob}\mathbf{u}) = (i\hbar/2m_o)(\psi^*\partial[\psi] - \partial[\psi^*]\psi) = \rho_{prob\_o}\mathbf{U} = \mathbf{J}/q$ $[\#/(m^2 \cdot s) = (\#/m^3)\cdot(m/s) = \#\text{-flux}]$
Particle # Conservation $(\partial \cdot \mathbf{N}) = 0$ → Closed System Total Particle # [N] = constant
Charge Conservation $(\partial \cdot \mathbf{J}) = 0$ → Closed System Total Charge [Q] = constant
Probability Conservation $(\partial \cdot \mathbf{J}_{prob}) = 0$ → Closed System Total Probability $\mathrm{Sum}[(\psi^*\psi)] = 1$ = constant

4-Vector **N** has dimensional units of [#-flux] and the 4-Scalar rest-number-density (n_o) has dimensional units of [#/volume].
4-Vector $\mathbf{J}_{prob}$ has dimensional units of [#-flux] and the 4-Scalar rest-probability-density (ρ_{prob_o}) has dimensional units of [#/volume].
This leads to the idea that the QM 4-ProbabilityCurrent $\mathbf{J}_{prob}$ is equivalent to the SR 4-(Dust)NumberFlux $\mathbf{N} = \mathbf{J}/q = \mathbf{J}_{prob}$.
The concepts are actually quite similar if one considers the fluid approximation of individual particles and generalized SR fluxes.
The fluid analogy allows densities that are less than unity, much as probabilities of expected positions of particles are less than unity and only sum to unity over the entire volume.
This argument is further strengthened by noting that in QM one also has 4-CurrentDensity $\mathbf{J} = q\mathbf{J}_{prob}$ (see again SR's $\mathbf{J} = q\mathbf{N}$)

GR → SR → RQM → QM → CM Classical Correspondence Principle:
In SR, one finds the Newtonian classical limiting-case approximation by using $\{|\mathbf{v}| \ll c\}$. In QM, there have been a variety of approaches to the Newtonian classical limiting-case approximation, including the idea of {number of particles ≫ 1}, the physics action much greater than the quantum unit $\{S \gg \hbar\}$, divergence small compared to system magnitude $\{\hbar|\nabla\cdot\mathbf{p}| \ll (\mathbf{p}\cdot\mathbf{p})\}$, etc. In the standard view of the theories of relativity and quantum mechanics, it is interesting to speculate on how the two "different" theories "conspire" to end up at the same classical mechanics physics as an approximation. However, in the SRQM view, this difficulty disappears. SR leads to RQM via the approach that has been shown. RQM then goes to QM as a limiting-case approximation by using $\{|\mathbf{v}| \ll c\}$. QM then goes to CM as a limiting-case in its own manner. There is a single chain of relationships, rather than two different theories "amazingly" approaching the same classical limit-case.

GR → {limit-case $g^{\mu\nu} \to \eta^{\mu\nu}$} → SR → derives → RQM → {limit-case $|\mathbf{v}| \ll c$} → QM → {limit-case $\hbar|\nabla\cdot\mathbf{p}| \ll (\mathbf{p}\cdot\mathbf{p})$} → (EM & CM)

Standard Postulates of QM (i.e. the mathematical structure of quantum mechanics), which SRQM must provide:
I) Description of the state of a system: The state of an isolated physical system is represented, at a fixed time t, by a state vector $| \psi \rangle$ belonging to a Hilbert space $\mathcal{H}$ called the state space: a vector space with an inner product operation (measurement & orthogonality).
II) Description of physical quantities: Every measurable physical quantity A is described by a Hermitian operator A acting in the state space $\mathcal{H}$. This operator is an observable, meaning that its eigenvectors form a basis for $\mathcal{H}$.
III) Measurement of physical quantities: The result of measuring a physical quantity A must be one of the eigenvalues of the corresponding observable A.
IV) Measurement of physical quantities: When the physical quantity A is measured on a system in a normalized state $| \psi \rangle$, the probability of obtaining an eigenvalue (denoted a_n for discrete spectra and α for continuous spectra) of the corresponding observable A is given by amplitude squared of the appropriate wave function (projection onto corresponding eigenvector).
V) Effect of measurement on the state: If the measurement of the physical quantity A on the system in the state $| \psi \rangle$ gives the result a_n, then the state of the system immediately after the measurement is the normalized projection of $| \psi \rangle$ onto the eigen-subspace associated with a_n
VI) Time evolution of a system: The time evolution of the state vector $| \psi(t) \rangle$ is governed by the Schrödinger relation, where H[t] is the observable associated with the total energy of the system (the Hamiltonian): $i\hbar\, d/dt\, | \psi(t) \rangle = H[t]\, | \psi(t) \rangle$

Analysis of the Koopman-von Neuman (KvN) formalism, a Hilbert Space framework which can give QM or CM:
The idea of Hilbert Space as requiring a quantum axiom is disproved by the Koopman–von Neumann formulation of classical mechanics, in which Hilbert Space mathematical formulation is successfully applied and results in the classical Liouville equation. This shows that the Hilbert Space framework is purely mathematical and can be applied to either/both classical and quantum systems. The main difference between which system emerges is the commutation relation between position and momentum. In the classical case, one assumes a zero-valued commutation relation. In the quantum case, there is a non-zero-valued commutation relation.
SR, as shown above, gives a non-zero commutation relation, thus leading naturally to the QM case. One instead uses a limiting-case approximation to go from QM to CM, in the same way that there is a limiting-case approximation to go from GR to SR, RQM to QM.

From Wikipedia on Koopman–von Neumann formulation (with some explanatory modifications):
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Derivation starting from operator axioms: It is possible to start from mathematical operator postulates, similar to the Hilbert space axioms of quantum mechanics, and derive the equation of motion by specifying how expectation values evolve.

The relevant axioms are that as in QM: (i) the states of a system are represented by normalized vectors of a complex Hilbert space $\mathcal{H}$, and the observables are given by self-adjoint operators acting on that space, (ii) the expectation value of an observable is obtained in the manner as the expectation value in quantum mechanics, (iii) the probabilities of measuring certain values of some observables are calculated by a "Born rule", and (iv) the state space of a composite system is the tensor product of the subsystem's spaces.

Mathematical form of the operator axioms: The above axioms (i) to (iv), written in the ⟨bra|,|ket⟩ notation, are
(i) $\langle \psi(t) | \psi(t) \rangle = 1$
(ii) The expectation value of an observable $\hat{A}$ at time t is $\langle A(t) \rangle = \langle \Psi(t) | \hat{A} | \Psi(t) \rangle$
(iii) The probability that a measurement of an observable $\hat{A}$ at time t yields A is $| \langle A | \Psi(t) \rangle |^2$, with $\hat{A} | A \rangle = A | A \rangle$.
(This axiom is an analogue of the Born rule in quantum mechanics.)
(iv) (see Tensor product of Hilbert spaces).

These axioms allow us to recover the formalism of both classical and quantum mechanics. Specifically, under the assumption that the classical position and momentum operators commute, the Liouville equation for the KvN wavefunction is recovered from averaged Newton's laws of motion. However, if the coordinate and momentum obey the non-zero canonical commutation relation, then the Schrödinger equation of quantum mechanics is obtained.

Measurements: In the Hilbert space and operator formulation of classical mechanics, the Koopman von Neumann–wavefunction takes the form of a superposition of eigenstates, and measurement collapses the KvN wavefunction to the eigenstate which is associated the measurement result, in analogy to the wave function collapse of quantum mechanics
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Thus, Hilbert Space is not a "quantum" axiom. Instead, Hilbert Space and its properties are a set of "mathematical" axioms and formulations, independent of the physics.
The formulation can be used by CM, and along with its zero commutation axiom, give classical Newtonian (CM) results.
The formulation can be used by QM, and along with its non-zero commutation axiom, give quantum (QM) results.
The formulation can be used by SRQM, and along with its non-zero commutation axiom, give relativistic quantum (RQM) results.

That the Hilbert Space axioms can be used in SRQM is the result of SRQM providing a linear PDE { $(\partial\cdot\partial) = -(m_o c/\hbar)^2$ }, which may be solved by generic mathematical Hilbert Space methods. Therefore, they are not quantum axioms, but emergent QM principles.

SR Derivation of CPT Symmetry:
The Lorentz Transformations $\Lambda^{\mu}{}_{\nu}$ play a fundamental role in SR. They describe the inherent symmetries of spacetime. The main ones usually mentioned are the continuous transforms, which depend on a set of parameters and are Proper {Det[continuous $\Lambda^{\mu}{}_{\nu}$] = +1}:
The temporal-spatial (B)oost $\Lambda^{\mu}{}_{\nu} \rightarrow B^{\mu}{}_{\nu}$ [$\boldsymbol{\beta}$] or [$\varphi,\hat{\mathbf{n}}$], 3 parameters, uses hyperbolic angles φ {cosh, sinh}, alternately relativistic $\boldsymbol{\beta}$ & γ.
The spatial-spatial (R)otation $\Lambda^{\mu}{}_{\nu} \rightarrow R^{\mu}{}_{\nu}$ [$\theta,\hat{\mathbf{n}}$], 3 parameters, uses the regular angles θ {cos, sin} about some axis $\hat{\mathbf{n}}$.

However, there also exist discrete Lorentz Transforms. One is the 4D Identity $\Lambda^{\mu}{}_{\nu} \rightarrow \delta^{\mu}{}_{\nu} = +I_{[4]}$, which leaves a system completely unchanged. It is a special case of both boost and rotation transforms with their parameters set to zero. $B^{\mu}{}_{\nu}$ [$|\boldsymbol{\beta}|=\mathbf{0}$] = $R^{\mu}{}_{\nu}$ [$\theta=0,\hat{\mathbf{n}}$] = $\delta^{\mu}{}_{\nu}$.

Most well known of the discrete Lorentz Transforms are the space-reversal (P)arity $\Lambda^{\mu}{}_{\nu} \rightarrow P^{\mu}{}_{\nu}$, which reverses the three spatial coordinates {$\mathbf{x} \rightarrow -\mathbf{x}$}, and the (T)imeReversal $\Lambda^{\mu}{}_{\nu} \rightarrow T^{\mu}{}_{\nu}$, which reverses the single temporal coordinate {$t \rightarrow -t$}. Less well known are the other Lorentz Transforms which include rotations of a fixed amount and spatial flips. It turns out that one can individually reverse any combination of the standard coordinates and still have a Lorentz Transform {essentially anything that has Det[$\Lambda^{\mu}{}_{\nu}$] = ±1}. Reversal of all the coordinates, {$t \rightarrow -t$} & {$\mathbf{x} \rightarrow -\mathbf{x}$} = {$\mathbf{X} \rightarrow -\mathbf{X}$}, gives the transform Combo(PT) $\Lambda^{\mu}{}_{\nu} \rightarrow (PT)^{\mu}{}_{\nu} = C^{\mu}{}_{\nu}$: proper trans. Examination of all possible combinations of Discrete Lorentz Transformations leads to CPT Symmetry:Invariance.

In other words, one can go from the Identity Transform $+I_{[4]}$ (all +1) to the Negative Identity Transform $-I_{[4]}$ (all -1) by doing a Combo(PT) = (C)hargeReversal Lorentz Transform. This Negative Identity has the interpretation of AntiMatter, without any need of Dirac's formulation using RQM. The Feynman-Stueckelberg AntiMatter Interpretation ~ CPT Interpretation (AntiMatter moving spacetime-backward = NormalMatter moving spacetime-forward) aligns with this.

Two interesting properties of (1,1)-Tensors, of which the Lorentz Transform is an example:
Mathematical Trace = Tr[..] = Sum (Σ) of EigenValues :
Mathematical Determinant = Det[..] = Product (Π) of EigenValues
As 4D (1,1)-Tensors, each Lorentz Transform $\Lambda^{\mu}{}_{\nu}$ has 4 EigenValues (EV's).
Create an Anti-Transform which has all EigenValue Tensor Invariants negated.
Σ[-(EV's)] = -Σ[EV's]: The Anti-Transform has negative Trace of the Transform.
Π[-(EV's)] = $(-1)^4$ Π[EV's] = Π[EV's]: The Anti-Transform has equal Determinant.
The Trace Invariant identifies a "Dual" Negative-Side for all Lorentz Transforms, with Proper Transform=Proper Anti-Transform.

This {NM= NormalMatter , AM= AntiMatter } interpretation can be analyzed using tensor determinant and trace operations.

Tr[NM-Rotate] = {0...+4}	Tr[NM-Identity] = +4	Tr[NM-Boost] = {+4...+∞}
Tr[AM-Rotate] = {0...−4}	Tr[AM-Identity] = −4	Tr[AM-Boost] = {−4...−∞}

t	x	y	z	Discrete **Normal Matter** (NM) Lorentz Transform Type	Trace	Determinant
+1	+1	+1	+1	NM-Minkowksi 4D Identity : AM-Flip-txyz=AM-Combo(PT)	Tr = +4	Det = +1 Proper
+1	+1	+1	–1	NM-Flip-z	Tr = +2	Det = –1 Improper
+1	+1	–1	+1	NM-Flip-y	Tr = +2	Det = –1 Improper
+1	+1	–1	–1	NM-Flip-yz=NM-Rotate-yz(π)	Tr = 0	Det = +1 Proper
+1	–1	+1	+1	NM-Flip-x	Tr = +2	Det = –1 Improper
+1	–1	+1	–1	NM-Flip-xz=NM-Rotate-xz(π)	Tr = 0	Det = +1 Proper
+1	–1	–1	+1	NM-Flip-xy=NM-Rotate-xy(π)	Tr = 0	Det = +1 Proper
+1	–1	–1	–1	NM-Flip-xyz=NM-ParityInverse:AM-Flip-t=AM-TimeReversal	Tr = –2	Det = –1 Improper
–1	+1	+1	+1	AM-Flip-xyz=AM-ParityInverse:NM-Flip-t=NM-TimeReversal	Tr = +2	Det = –1 Improper
–1	+1	+1	–1	AM-Flip-xy=AM-Rotate-xy(π)	Tr = 0	Det = +1 Proper
–1	+1	–1	+1	AM-Flip-xz=AM-Rotate-xz(π)	Tr = 0	Det = +1 Proper
–1	+1	–1	–1	AM-Flip-x	Tr = –2	Det = –1 Improper
–1	–1	+1	+1	AM-Flip-yz=AM-Rotate-yz(π)	Tr = 0	Det = +1 Proper
–1	–1	+1	–1	AM-Flip-y	Tr = –2	Det = –1 Improper
–1	–1	–1	+1	AM-Flip-z	Tr = –2	Det = –1 Improper
–1	–1	–1	–1	AM-Minkowksi 4D Identity : NM-Flip-txyz=NM-Combo(PT)	Tr = –4	Det = +1 Proper
$\bar{t}$	$\bar{x}$	$\bar{y}$	$\bar{z}$	Discrete **AntiMatter** (AM) Lorentz Transform Type	Trace	Determinant

There is complete (+/-) symmetry, which agrees with all known experiments with NormalMatter ←☯→ AntiMatter to-date.

Grouped and ordered by the trace values, one gets:

Discrete **Normal Matter** (NM) Lorentz Transform Type	Trace	Determinant
NM-Minkowksi 4D Identity $+I_{[4]}$ AM-Flip-txyz=AM-Combo(PT)=AM-NegateIdentity=AM-NegateCharge	Tr = +4	Det = +1 Proper
NM-Flip-t ,NM-Flip-x, NM-Flip-y, NM-Flip-z AM-Flip-xyz=AM-ParityInverse	Tr = +2	Det = –1 Improper
NM-Flip-xy=NM-Rotate-xy(π), NM-Flip-xz=NM-Rotate-xz(π), NM-Flip-yz=NM-Rotate-yz(π) AM-Flip-xy=AM-Rotate-xy(π), AM-Flip-xz=AM-Rotate-xz(π), AM-Flip-yz=AM-Rotate-yz(π)	Tr = 0	Det = +1 Proper
NM-Flip-xyz=NM-ParityInverse AM-Flip-t ,AM-Flip-x, AM-Flip-y, AM-Flip-z	Tr = –2	Det = –1 Improper
NM-Flip-txyz=NM-Combo(PT)=NM-NegateIdentity=NM-NegateCharge AM-Minkowksi 4D Identity $-I_{[4]}$	Tr = –4	Det = +1 Proper
Discrete **AntiMatter** (AM) Lorentz Transform Type	Trace	Determinant

This clearly shows that Combo(PT) Transform is equivalent to a (C)harge Transform, which flips NormalMatter ←☯→ AntiMatter. Also, this (C)harge Transform is Proper, with a determinant of +1, the same as the Boost, Rotation, Flip-TwoCoords Transforms, which means that it can occur in reality. Overall, this is the source of (CPT) Symmetry.

SRQM Conclusion:

Using standard Tensor calculus (especially the 4-Vector formulations} of Einstein-Minkowski 4D Spacetime it is shown that foundational features of spacetime common to both Special Relativity (SR) and Quantum Mechanics (QM) exist. Many fundamental physical relations are encoded in the simple tensorial rules of 4-Vectors and their Lorentz Scalar Products.

(ħ) is shown to be an empirically-measurable fundamental physical constant and a Lorentz 4-Scalar Invariant, just like LightSpeed (c).

The 4-Vector relations $\mathbf{P} = m_o\mathbf{U}$ {particle view [·] ↠} : $\mathbf{K} = (\omega_o/c^2)\mathbf{U}$ {wave view [⌇] ⟿} : $\mathbf{P} = \hbar\mathbf{K}$ {particle-wave view [·⌇]} are all shown to be isomorphic in the sense that all are derivable from SR and the rules of Tensor calculus.

The mathematical relation $\mathbf{K} = i\partial$ or $\partial = -i\mathbf{K}$ and the existence of a complex wavefunction Ψ is shown applicable to all types of waves: classical, quantum, relativistic, EM, purely-mathematical. 4-Vector **K** is a solution of the 4D Invariant d'Alembertian Wave Eqn. $(\partial\cdot\partial)$.

All waves are described by 4D Tensor amplitudes $(a)=\{A,A^\mu,A^{\mu\nu},etc.\}$ and the Lorentz Scalar Product function $e^{\pm i(\mathbf{K}\cdot\mathbf{X})}$ propagator.

The combination of these relations lead to a KG relativistic quantum wave relation $(\partial\cdot\partial) = (im_oc/\hbar)^2 = -(m_oc/\hbar)^2 = -(\omega_o/c)^2$, i.e. the invariant d'Alembertian, and to the 4-Vector form of the standard Schrödinger relations $\mathbf{P} = i\hbar\partial$, which give $\{E = i\hbar\partial_t\}$ and $\{\mathbf{p} = -i\hbar\nabla\}$

There exists a non-zero commutation relation in SR: $[X^\mu, P^\nu] = -i\hbar\eta^{\mu\nu}$. which gives standard canonical QM commutation $[x^j, p^k] = i\hbar\delta^{jk}$

There exists a conserved current $\mathbf{J}_{prob}$ in SR, with $(\partial\cdot\mathbf{J}_{\mathbf{prob}}) = 0$, based on Green's vector identity applied to the KG relation.

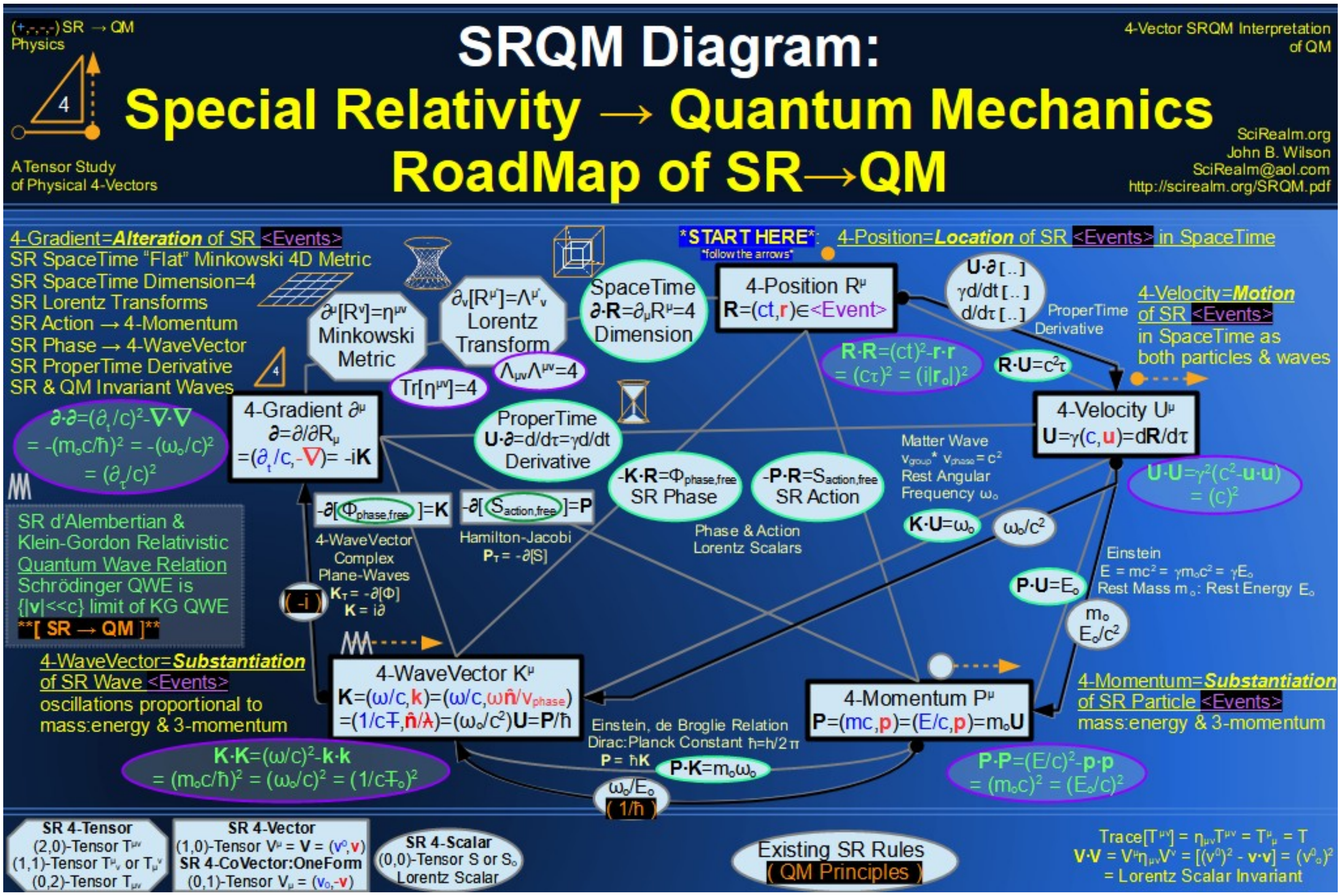
The SR 4-(Dust)NumberFlux **N** appears to be equivalent to the RQM 4-ProbabilityCurrentDensity $\mathbf{J}_{\mathbf{prob}}$.

The standard Born probability interpretation, $(\psi^* \psi) = \rho_{prob\_o}$, emerges in the low-potential-energy and low-velocity limit.

CPT Symmetry emerges from an analysis of the mathematical properties of the Lorentz Transformations $\Lambda^\mu{}_\nu$ and particularly from the Invariant Tensor Trace $Tr[\Lambda^\mu{}_\nu]$.

The correspondence principle of both SR and QM to Newtonian classical physics CM is discussed.
GR → {limit-case $g^{\mu\nu} \to \eta^{\mu\nu}$} → SR → derives → RQM → {limit-case $|\mathbf{v}| \ll c$} → QM → {limit-case $\hbar|\nabla\cdot\mathbf{p}| \ll (\mathbf{p}\cdot\mathbf{p})$} → (EM & CM)

One can derive the "axioms" of QM from the principles of SR, hence [SR→QM].



Start with a few SR Physical 4-Vectors:

4-Position	$\mathbf{R} = (ct,\mathbf{r})$
4-Velocity	$\mathbf{U} = \gamma(c,\mathbf{u})$
4-Momentum	$\mathbf{P} = (E/c,\mathbf{p}) = (mc,\mathbf{p})$
4-WaveVector	$\mathbf{K} = (\omega/c,\mathbf{k})$
4-Gradient	$\partial = (\partial_t/c,-\nabla)$

Note the following relations between SR 4-Vectors:

$\mathbf{U} = d\mathbf{R}/d\tau$

$\mathbf{P} = m_o\mathbf{U} = (E_o/c^2)\mathbf{U}$ [·] {particle motion}

$\mathbf{K} = (1/\hbar)\mathbf{P} = (\omega_o/c^2)\mathbf{U}$ [⌇] {wave motion}

$\partial = -i\mathbf{K}$

Form a chain of SR Lorentz Invariant Scalar Equations, based on those relations:

$\mathbf{R}\cdot\mathbf{R} = (c\tau)^2$

$\mathbf{U}\cdot\mathbf{U} = (c)^2$

$\mathbf{P}\cdot\mathbf{P} = (m_oc)^2 = (E_o/c)^2$

$\mathbf{K}\cdot\mathbf{K} = (m_oc/\hbar)^2 = (\omega_o/c)^2$

$\partial\cdot\partial = (-im_oc/\hbar)^2 = -(m_oc/\hbar)^2 = -(\omega_o/c)^2$

This is (RQM) = Relativistic Quantum Mechanics, derived from only:

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5 of the Standard SR 4-Vectors

4 really simple empirical relations between them

1 SR rule for forming Lorentz Scalar Invariants, i.e. the Minkowski Metric ($\eta_{\mu\nu}$) which gives the Lorentz Scalar Product (·)

4-Creativity ☼ = (Music,**Artwork**) : 4-Universality ∞ = (Eternity,**Infinity**) : 4-Origin **O** = (Now,**Here**)

Thus, this treatise explains (or at least gives a probable reason for) the following empirical observations:
Why GR works so well in its realm of applicability {massive large-scale systems}.
Why QM works so well in its realm of applicability {micro-scale systems and special macroscopic systems, ex. superfluids}.
 i.e. The tangent space to GR curvature at any point is locally Minkowskian, and thus QM works for small volumes...
Why RQM explains physical effects that QM-without-SR cannot, and with greater accuracy those that basic QM can explain.
Why attempts to "quantize gravity" fail {essentially, everyone has been trying to put the cart (QM) before the horse (GR)}.
Why all attempts to modify GR keep conflicting with experimental data {because to-date, GR is apparently still fundamental}.
Why QM works with SR as RQM, but not with GR {because QM is derivable from SR, hence a manifestation of SR rules}.
In other words, the "special case" rules of QM are not something that can be imposed on GR, which is the "general" parent theory.
How Minkowski Space, 4-Vectors, and Lorentz Invariants play vital roles in RQM, & give the SRQM Interpretation of QM.

Major clues from experiment, observation, and mathematics that is actually proven to be related to physical reality:
The components of 4-Vectors and 4-Tensors are experimentally measurable elements/properties of physical reality.
Both General Relativity (GR) and Special Relativity (SR) have passed very stringent tests of multiple varieties.
Relativistic Quantum Mechanics (RQM) and standard Quantum Mechanics (QM) have passed all tests within their realms of validity:
RQM describes and explains phenomena that standard QM cannot.
Mass **and** Spin are the Casimir Invariants of Poincaré Invariance, which comes from 4D SR:Minkowski Space, not QM.
 Hence, neither mass nor spin require a quantum axiom for their existence, yet are the only "quantum numbers" of physical states.
To-date, there is no observational/experimental indication that quantum effects "alter" the fundamentals of either SR or GR.
 There have been no repeatable violations of Poincaré-Lorentz Invariance, nor of Local Position Invariance, nor of CPT Symmetry,
 nor of the Standard Model formulation (neutrino masses notwithstanding). This rules out many alternative gravity theories.
In fact, in all known experiments where both SR/GR and QM are present, QM respects the principles of SR/GR,
 whereas SR/GR modify the results of QM, ex. Dirac RQM Wave Eqn. vs Schrödinger QM Wave Eqn.
All tested quantum-level particles, atoms, isotopes, molecules, super-positions, spin-states, excited-states, etc. obey:
 GR's {Universality of Free-Fall & Equivalence Principle} and SR's { $E = mc^2$ & lightspeed (c) communication/signaling limit}.
On the other hand, GR gravity *does* induce changes in quantum interference patterns and hence modifies QM.
For instance, quantum-level atomic-clocks are used to measure gravitational "Doppler" red:blue-shift effects.
 i.e. GR gravitational frequency-shift (gravity time-dilation) alters atomic=quantum-level timing. Think about that for a moment...
While quantum entanglement is somewhat mysterious, there is no FTL-communication-with nor alteration-of distant particles.
Getting a Stern-Gerlach "up" |here⟩ doesn't cause the distant entangled-particle to suddenly start physically moving "down" |there⟩.
The universe appears to be both: Homogeneous = same all places [▦](SpaceTime-Translation ΔX^μ Symmetry) over displacement $\Delta\mathbf{X}$
and Isotropic = same all directions (∗) (Lorentz $\Lambda^\mu{}_\nu$ Symmetry) over {angle θ, hyperbolic angle φ}.
The main Schrödinger relation is just a special case of complex plane waves on objects (4-Vectors) already connected in standard SR.
All Lorentz Scalar Product connections between SR 4-Vectors are Lorentz Invariants, and typically fundamental constants with,
 to-date, no consistent evidence of change over time during the age of the universe nor of varying value throughout spatial extent.
CM typically uses a Phase Space description; QM typically uses a Hilbert Space description. However:
 CM can be done with a Hilbert Space description: see Koopman-von Neumann Classical Mechanics.
 QM can be done without a Hilbert Space description: see Phase Space Formulation of Quantum Mechanics.
 Hence, Hilbert space is purely mathematical: it and its associated properties do not require a quantum axiom for its existence.
Particles of both CM and QM can be described by a wave-like theory: see the Hamilton-Jacobi Equations.
Measurement of Planck's constant (h) can be done with experiments that do not need quantum theory for the measurement, just SR.
In other words, the actual measurement process uses empirical, non-QM-theory-dependent components {don't need Schrödinger eqn}.
50+ years of unsuccessful attempts to "quantize gravity".
To-date, no new particles found outside of the Standard Model by LHC or other experiments, just the expected SM Higgs Boson.

Other SRQM sources – most current versions can be found at SciRealm.org:
Main website: SRQM - QM from SR - Simple RoadMap (.html)
PDF slideshow presentation/book: http://scirealm.org/SRQM.pdf
This Summary: http://scirealm.org/SRQM-Summary.pdf
Alternate discussion at: http://scirealm.org/SRQM.html
SRQM Flyer: http://scirealm.org/SRQM_Flyer.pdf
See: 4-Vectors & Lorentz Scalars Reference for more info on Four-Vectors (4-Vectors) in general
See: http://scirealm.org/SRQM-FundamentalConstants.pdf
See: John's Online RPN Scientific Calculator, using Complex Math
See: SRQM - Online SR 4-Vector & Tensor Calculator
https://www.researchgate.net/project/SR--QM-Special-Relativity--Quantum-Mechanics-Project
https://www.researchgate.net/project/Classical-Foundations-of-Quantum-Mechanics, with William P. Rice

Another way the study SR 4-Vectors and Invariant Lorentz Scalar relations between them, assume them as primitives:

4-Position	$\mathbf{R} = R^{\mu} = (ct,\mathbf{r}) \in$ <Event> $\in$ <Time·Space>
4-Velocity	$\mathbf{U} = U^{\mu} = \gamma(c,\mathbf{u}) = (\mathbf{U}\cdot\boldsymbol{\partial})\mathbf{R} = (d/d\tau)\mathbf{R}$
4-Momentum	$\mathbf{P} = P^{\mu} = (E/c,\mathbf{p}) = (m_o)\mathbf{U}$
4-WaveVector	$\mathbf{K} = K^{\mu} = (\omega/c,\mathbf{k}) = (1/\hbar)\mathbf{P}$
4-Gradient	$\boldsymbol{\partial} = \partial^{\mu} = (\partial_t/c,-\nabla) = (-i)\mathbf{K}$

Assume the 4-Vectors above, which are 4D (1,0)-Tensors, are primitive axioms.
Use the rules of Tensors to derive various other properties.

The mathematical Lorentz Scalar Product of two generic 4-Vectors $\mathbf{A} = A^{\mu} = (a^0,\mathbf{a})$ and $\mathbf{B} = B^{\mu} = (b^0,\mathbf{b})$ is:
$(\mathbf{A}\cdot\mathbf{B}) = A^{\mu}\eta_{\mu\nu}B^{\nu} = A_{\nu}B^{\nu} = A^{\mu}B_{\mu} = (a^0b^0 - \mathbf{a}\cdot\mathbf{b}) = (a^0{}_o b^0{}_o)$ which is a Lorentz Invariant "Rest" 4-Scalar { 4D (0,0)-Tensor }

$(\mathbf{U}\cdot\boldsymbol{\partial}) = \gamma[\partial_t + \mathbf{u}\cdot\nabla] = \gamma d/dt = d/d\tau$	: Proper Time Derivative $(d/d\tau)$	[1/s]
$(\mathbf{K}\cdot\mathbf{R}) = (\omega t) - \mathbf{k}\cdot\mathbf{r} = (\omega_o t_o) = -\Phi$	: Invariant SR WavePhase (Φ) or (Φ_{phase})	[{rad}]
$(\mathbf{P}\cdot\mathbf{R}) = (Et) - \mathbf{p}\cdot\mathbf{r} = (E_o t_o) = -S$	: Invariant SR Action (S) or (S_{action})	[$kg\cdot m^2/s = J\cdot s$ = Action]
$(\mathbf{K}\cdot\mathbf{U}) = \gamma[\omega - \mathbf{k}\cdot\mathbf{u}] = (\omega_o)$	: Invariant SR RestAngFreq (ω_o)	[{rad}/s]
$(\mathbf{P}\cdot\mathbf{U}) = \gamma[E - \mathbf{p}\cdot\mathbf{u}] = (E_o)$	: Invariant SR RestEnergy (E_o)	[$kg\cdot m^2/s^2 = J$]

One can divide two generic 4-Vectors $\mathbf{A} = A^{\mu} = (a^0,\mathbf{a})$ and $\mathbf{B} = B^{\mu} = (b^0,\mathbf{b})$ by using an intermediary 4-Vector $\mathbf{V} = V^{\mu} = (v^0,\mathbf{v})$:
$|\mathbf{A}|/|\mathbf{B}| = (\mathbf{V}\cdot\mathbf{A})/(\mathbf{V}\cdot\mathbf{B}) = (\mathbf{A}\cdot\mathbf{V})/(\mathbf{B}\cdot\mathbf{V}) = (a^0{}_o v^0{}_o)/(b^0{}_o v^0{}_o) = (a^0{}_o/b^0{}_o)$ which is a Lorentz Invariant 4-Scalar { 4D (0,0)-Tensor }

Let's use our primitive 4-Vectors from above to derive various Invariant Lorentz Scalars = 4D (0,0)-Tensors

$|\mathbf{R}|/|\mathbf{U}| = (\mathbf{U}\cdot\mathbf{R})/(\mathbf{U}\cdot\mathbf{U}) = (c^2t_o)/(c^2) = (t_o) = (\tau)$ = ProperTime:	$\mathbf{R} = (\tau)\mathbf{U}$ assuming constant $\mathbf{U}$, similar to 3D $\mathbf{r} = t^*\mathbf{v}$
$|\mathbf{P}|/|\mathbf{U}| = (\mathbf{U}\cdot\mathbf{P})/(\mathbf{U}\cdot\mathbf{U}) = (E_o)/(c^2) = (m_o)$ = RestMass:	$\mathbf{P} = (m_o)\mathbf{U}$
$|\mathbf{K}|/|\mathbf{U}| = (\mathbf{U}\cdot\mathbf{K})/(\mathbf{U}\cdot\mathbf{U}) = (\omega_o)/(c^2) = (\omega_o/c^2)$ = RestAngFreq/c^2:	$\mathbf{K} = (\omega_o/c^2)\mathbf{U}$
$|\mathbf{P}|/|\mathbf{K}| = (\mathbf{U}\cdot\mathbf{P})/(\mathbf{U}\cdot\mathbf{K}) = (E_o)/(\omega_o) = (\hbar)$ = PlanckConstant:	$\mathbf{P} = (\hbar)\mathbf{K}$
$|\boldsymbol{\partial}|/|\mathbf{K}| = (\mathbf{U}\cdot\boldsymbol{\partial})/(\mathbf{U}\cdot\mathbf{K}) = (d/d\tau)/(\omega_o) = (d/\omega_o d\tau)$ = ModProperTimeDifferential:	$\boldsymbol{\partial} = (\mathbf{K}/\omega_o)(d/d\tau) \rightarrow \boldsymbol{\partial} = (-i)\mathbf{K}$

$\boldsymbol{\partial}[\psi] = (\mathbf{K}/\omega_o)(d/d\tau)[\psi]$: a solution to this is $\psi = ae^{\wedge}\{b(\mathbf{K}\cdot\mathbf{R})\}$ with constant $\{a,b,\mathbf{K}\}$

$\boldsymbol{\partial}[\psi] = ae^{\wedge}\{b(\mathbf{K}\cdot\mathbf{R})\}\ \boldsymbol{\partial}[b(\mathbf{K}\cdot\mathbf{R})] = \psi\ \boldsymbol{\partial}[b(\mathbf{K}\cdot\mathbf{R})] = \psi\ (\partial/\partial\underline{\mathbf{R}})[b(\mathbf{K}\cdot\mathbf{R})] = \psi b\mathbf{K}$
$(d/d\tau)[\psi] = ae^{\wedge}\{b(\mathbf{K}\cdot\mathbf{R})\}\ (d/d\tau)[b(\mathbf{K}\cdot\mathbf{R})] = \psi\ (d/d\tau)[b(\mathbf{K}\cdot\mathbf{R})] = \psi\ [b(\mathbf{K}\cdot\mathbf{U})] = \psi\ [b(\omega_o)] = \psi b\omega_o$
$(\mathbf{K}/\omega_o)(d/d\tau)[\psi] = (\mathbf{K}/\omega_o)(\psi b\omega_o) = \psi b\mathbf{K}$

so, $\boldsymbol{\partial}[..] = [..]b\mathbf{K}$
To get a solution that doesn't fall to zero {real $b\omega_o<1$} or blow up to infinity {real $b\omega_o>1$}, we can choose $b = \pm i$
so, common choice is { $\boldsymbol{\partial} = (-i)\mathbf{K}$ } or { $\mathbf{K} = i\boldsymbol{\partial}$ }

Note the following relations between SR 4-Vectors:
$\mathbf{U} = d\mathbf{R}/d\tau$ $\quad\quad$ $\mathbf{R} = (\tau)\mathbf{U}$ assuming constant $\mathbf{U}$: $d\mathbf{R}/d\tau = d(\tau)\mathbf{U}/d\tau = \mathbf{U}$
$\mathbf{P} = m_o\mathbf{U} = (E_o/c^2)\mathbf{U}$
$\mathbf{K} = (1/\hbar)\mathbf{P} = (\omega_o/c^2)\mathbf{U}$
$\boldsymbol{\partial} = -i\mathbf{K}$

Form a chain of SR Lorentz Invariant Scalar Equations, based on those relations:
$\mathbf{R}\cdot\mathbf{R} = (c\tau)^2$
$\mathbf{U}\cdot\mathbf{U} = (c)^2$
$\mathbf{P}\cdot\mathbf{P} = (m_oc)^2 = (E_o/c)^2$
$\mathbf{K}\cdot\mathbf{K} = (m_oc/\hbar)^2 = (\omega_o/c)^2$
$\boldsymbol{\partial}\cdot\boldsymbol{\partial} = (-im_oc/\hbar)^2 = -(m_oc/\hbar)^2 = -(\omega_o/c)^2$

The last equation is the Klein-Gordon (RQM) Wave Equation = Relativistic Quantum Mechanics.
Also, again note that { $\mathbf{P} = (\hbar)\mathbf{K}$ } and { $\mathbf{K} = (i)\boldsymbol{\partial}$ } combined give { $\mathbf{P} = (i\hbar)\boldsymbol{\partial}$ } the Schrödinger Relation.

Relativistic Thermodynamics (used to derive the Ideal Gas Law, using the Perfect Fluid StressEnergy Tensor):

4-Position	$\mathbf{R} = R^\mu = (ct,\mathbf{r})$		
4-Velocity	$\mathbf{U} = U^\mu = \gamma(c,\mathbf{u})$	$= d\mathbf{R}/d\tau$	τ is ProperTime or RestTime
4-Acceleration	$\mathbf{A} = A^\mu = \gamma(c\gamma',\gamma'\mathbf{u}+\gamma\mathbf{a})$	$= d\mathbf{U}/d\tau = d^2\mathbf{R}/d\tau^2$	
4-Momentum	$\mathbf{P} = P^\mu = (mc,\mathbf{p}=m\mathbf{u}) = (E/c,\mathbf{p})$	$= (m_o)\mathbf{U}$	m_o is RestMass
4-"Dust"NumberFlux	$\mathbf{N} = N^\mu = (nc,\mathbf{n}=n\mathbf{u})$	$= (n_o)\mathbf{U}$	n_o is RestNumberDensity
4-EntropyFlux	$\mathbf{S} = S^\mu = (sc,\mathbf{s}=s\mathbf{u})$	$= (s_o)\mathbf{U}$	s_o is RestEntropyDensity
4-ThermalVector	$\mathbf{\Theta} = \Theta^\mu = (\theta^0,\boldsymbol{\theta}) = (c/[k_BT],\mathbf{u}/[k_BT]) = (\theta_o/c = \beta_o = 1/[k_BT_o])\mathbf{U}$		= 4-InverseTemperatureMomentum Thermodynamic Rest $\beta_o = 1/[k_BT_o]$

Minkowski Metric 4D (0,2)-Tensor format	$\eta_{\mu\nu} = V_{\mu\nu} + H_{\mu\nu} \rightarrow \mathrm{Diag}[1,-1,-1,-1]_{\mathrm{Cartesian}}$
Temporal-Projection 4D (0,2)-Tensor format	$V_{\mu\nu} \rightarrow \mathrm{Diag}[1,0,0,0]_{\mathrm{Cartesian}}$
Spatial-Projection 4D (0,2)-Tensor format	$H_{\mu\nu} \rightarrow \mathrm{Diag}[0,-1,-1,-1]_{\mathrm{Cartesian}}$
PerfectFluid StressEnergy 4D (2,0)-Tensor format	$T^{\mu\nu} \rightarrow \mathrm{Diag}[\rho_e,p,p,p]_{\mathrm{MCRF}}$: ρ_e=energy_density : p=spatial pressure $[J/m^3=N/m^2]$

Thermodynamic Relations:

EnergyDensity	$\rho_{eo} = V_{\mu\nu}T^{\mu\nu}$	: EnergyDensity = Temporal Projection of the PerfectFluid StressEnergy
Pressure (tensor comp)	$p = p_o = (-1/3)H_{\mu\nu}T^{\mu\nu}$	: Pressure = Spatial Projection of the PerfectFluid StressEnergy
Pressure	$P = P_o$	
Volume	$V = V_o / \gamma$	
EntropyDensity	$s = \gamma s_o$	
Entropy	$S = S_o$	use $S = S_o$ for Entropy, $s = \gamma s_o = S/V$ for entropy density
Temperature	$T = T_o / \gamma$	
NumberDensity	$n = \gamma n_o$	
Number Particles	$N = nV = (\gamma n_o)(V_o / \gamma) = n_oV_o = N_o$	
Number (Amount)	$N = N_o$	use $N = N_o$ for Number, $n = \gamma n_o = N/V$ for number density
ChemicalPotential	$\mu = \mu_o / \gamma$	
Heat	$Q = Q_o/\gamma$	
Work	$W = W_o \gamma$	

{ Pressure $p=p_o$: Number of Particles $N=N_o$: Entropy $S=S_o$: Boltzmann Constant (k_B) } are all Lorentz Scalars

Note, these come in "energy paired-components".

(Entropy)*(Temperature)	:	(Pressure)*(Volume)	:	(Number)*(Chemical Potential)
$(S = S_o)*(T = T_o/\gamma)$	:	$(P = P_o)*(V = V_o/\gamma)$	:	$(N = N_o)*(\mu = \mu_o/\gamma)$
(extensive):(intensive)	:	(intensive):(extensive)	:	(extensive):(intensive)
(**invariant**):(*relativistic*)	:	(**invariant**):(*relativistic*)	:	(**invariant**):(*relativistic*)

It is curious as to why the extensive*intensive don't exactly match the **invariant**:*relativistic* in this form.
Extensive means the value changes with system size. Intensive means the value is independent of changes in system size or amount.
ex. Expose two separate systems, one big, one small, to an external bath. Then, close each. Both of the now-closed systems will have the same temperature, but the small system will have less entropy than the big one. Both will have the same pressure, but the small will have less volume. Both will have the same chemical potential, but the small will have fewer particles,

One can divide everything by *relativistic* volume V to make everything intensive and overall **invariant**.

$(s = S/V)*(T = T_o/\gamma)$	:	$(P = P_o)*(V/V)$	:	$(n = N/V)*(\mu = \mu_o/\gamma)$
$(s = \gamma s_o)*(T = T_o/\gamma)$	:	$(P = P_o)$	:	$(n = \gamma n_o)*(\mu = \mu_o/\gamma)$
$(sT = s_oT_o)$	:	$(P = P_o)$	:	$(n\mu = n_o\mu_o)$

The energy density e=E/V of a system ~ $sT - P + n\mu$

The 4-ThermalVector $\mathbf{\Theta}$ is used in Relativistic Thermodynamics.
It is seen in the calculation of thermal distributions: $\mathbf{P}\cdot\mathbf{\Theta} = (E/c,\mathbf{p})\cdot(c/[k_BT],\mathbf{u}/[k_BT]) = E/[k_BT] - \mathbf{p}\cdot\mathbf{u}/[k_BT] = E_o/[k_BT_o]$
This is a dimensionless Lorentz invariant used in the Boltzmann factor of partition functions of SM: $e^{-(E_o/[k_BT_o])} = e^{-(\beta_oE_o)}$

The definition of the 4-ThermalVector matches the original definition of the Planck-Einstein temperature transformation rule.
4-ThermalVector $\mathbf{\Theta} = (c/[k_BT],\mathbf{u}/[k_BT]) = (1/[k_BT_o])\mathbf{U} = (1/[k_BT_o])\gamma(c,\mathbf{u}) = \gamma(c/[k_BT_o],\mathbf{u}/[k_BT_o])$
which gives $(1/T) = \gamma(1/T_o)$ or $T = T_o/\gamma$. The temperature transforms like a volume, $V = V_o/\gamma$.

Relativistic Gibbs-Duhem eqn:
$\mathbf{U}\cdot\mathbf{P} = \gamma(E-\mathbf{u}\cdot\mathbf{p}) = E_o = (T_oS_o - P_oV_o + \mu_oN_o) = m_oc^2$ for a spatially homogeneous system

Manifestly-Invariant-Tensor-Forms (using only 4D Tensors: 4-Vectors & a Lorentz Scalar)

$(p_o)\mathbf{\Theta} = \mathbf{N}$ $[(N/m^2)\cdot(1/N\cdot s) = (\#/m^2\cdot s)]$ The Lorentz Scalar Pressure * 4-ThermalVector = 4-"Dust"NumberFlux

$(p_o)(1/[k_B T_o])\mathbf{U} = (n_o)\mathbf{U}$

$(p_o/[k_B T_o]) = (n_o)$

$(p_o/[k_B T_o]) = (N_o/V_o)$

$p_o V_o = N_o k_B T_o$

$p_o V_o/\gamma = N_o k_B T_o/\gamma$

$(p_o)(V_o/\gamma) = (N_o)(k_B)(T_o/\gamma)$

$(p)(V) = (N)(k_B)(T)$ because Lorentz Scalars: $\{ p = p_o : N = N_o : (k_B)$ is a physical constant $\}$

$pV = Nk_B T$ $[(N/m^2)\cdot(m^3) = (N\cdot m) = J]$

Ideal Gas Law in Standard Form

$(\rho_{eo})\mathbf{\Theta} = (f)\mathbf{N}$ $[(J/m^3)\cdot(m/J\cdot s) = (\#/m^2\cdot s)]$

$(\rho_{eo})(1/[k_B T_o])\mathbf{U} = (f)(n_o)\mathbf{U}$

$(\rho_{eo})(1/[k_B T_o]) = (f)(n_o)$

$(E_o/V_o)(1/[k_B T_o]) = (f)(N_o/V_o)$

$(E_o)(1/[k_B T_o]) = (f)(N_o)$

$E_o = (f)N_o k_B T_o$

$E_o = (f)N_o k_B T_o$

$(\gamma/\gamma)E_o = (f)N_o k_B(\gamma/\gamma)T_o$

$(1/\gamma)E = (f)N_o k_B(\gamma/1)T$

$E = \gamma^2(f)N_o k_B T$

$E = \gamma^2(f)(N)(k_B)(T)$

$E \rightarrow (f)Nk_B T$ for $\{ v \ll c \}$

For a monatomic ideal gas, $(f) = 3/2$

$E = (3/2)Nk_B T$ $[J]$

Energy Equation of State for Ideal Gas

4-(EM)VectorPotential	$\mathbf{A} = A^\mu = (\varphi/c,\mathbf{a})$	$[kg\cdot m/(C\cdot s) = T\cdot m]$
4-Momentum	$\mathbf{P} = P^\mu = (mc,\mathbf{p}=m\mathbf{u}) = (E/c,\mathbf{p})$	$[kg\cdot m/s]$
4-PotentialMomentum	$\mathbf{Q} = Q^\mu = (q\varphi/c,q\mathbf{a}) = q\mathbf{A}$	$[kg\cdot m/s]$
4-ThermalVector	$\mathbf{\Theta} = \Theta^\mu = (\theta^0,\boldsymbol{\theta}) = (c/[k_B T],\mathbf{u}/[k_B T]) = (\theta_o/c = \beta_o = 1/[k_B T_o])\mathbf{U}$	$[s/kg\cdot m]$

$\mathbf{Q}\cdot\mathbf{\Theta} = (q\varphi/c,q\mathbf{a})\cdot(c/[k_B T],\mathbf{u}/[k_B T]) = (q\varphi/[k_B T] - q\mathbf{a}\cdot\mathbf{u}/[k_B T]) = (q\varphi_o/[k_B T_o]) = 1$

For an electronic thermal voltage, one obtains $q\varphi_o = k_B T_o$: with $q = |e|$

Essentially, one trades electrical energy for thermal energy.

$\mathbf{P}\cdot\mathbf{\Theta} = (E/c,\mathbf{p})\cdot(c/[k_B T],\mathbf{u}/[k_B T]) = (E/[k_B T] - \mathbf{p}\cdot\mathbf{u}/[k_B T]) = (E_o/[k_B T_o])$

The Boltzmann Probability Distribution Factor of Statistical Mechanics

The Partition Function of SM is $Z = \Sigma_i\, [e\text{^}{-}\mathbf{P}_i\cdot\mathbf{\Theta}\,]$

Relativistic Motion: {4D General, 3D Circular=4D TimeHelix, 4D Hyperbolic, 4D Linear}

General SR Equations:

$\mathbf{R}^{n\prime} = d\mathbf{R}^{(n-1)\prime}/d\tau = d^{(n-1)}\mathbf{R}/d\tau^{(n-1)}$ $\mathbf{J} = \mathbf{A}' = \mathbf{U}'' = \mathbf{R}'''$ $' = d/d\tau = \gamma d/dt$

$\mathbf{r}^{n\prime} = d\mathbf{r}^{(n-1)\prime}/dt = d^{(n-1)}\mathbf{r}/dt^{(n-1)}$ $\mathbf{j} = \mathbf{a}' = \mathbf{u}'' = \mathbf{r}'''$ $' = d/dt$

[SR Degrees of Freedom (DoF)] + [# of Constraints] = 10 due to Poincaré Group

$(\mathbf{U}\cdot\mathbf{U}) = c^2$ is temporal, invariant, fundamental constant

$d/d\tau[\mathbf{U}\cdot\mathbf{U}] = d/d\tau[c^2] = 0 = d/d\tau[\mathbf{U}]\cdot\mathbf{U} + \mathbf{U}\cdot d/d\tau[\mathbf{U}] = 2(\mathbf{A}\cdot\mathbf{U}) = 0$

$(\mathbf{A}\cdot\mathbf{U} = 0) \leftrightarrow (\mathbf{A}\perp\mathbf{U})$: 4-Acceleration (normal to worldline) is orthogonal($\perp$) to 4-Velocity (tangent to worldline)

$d/d\tau[\mathbf{A}\cdot\mathbf{U}] = 0 = d/d\tau[\mathbf{A}]\cdot\mathbf{U} + \mathbf{A}\cdot d/d\tau[\mathbf{U}] = \mathbf{J}\cdot\mathbf{U} + \mathbf{A}\cdot\mathbf{A} = \mathbf{J}\cdot\mathbf{U} + -(\alpha)^2$

$(\mathbf{J}\cdot\mathbf{U}) = (\alpha)^2 = -(\mathbf{A}\cdot\mathbf{A})$

For a particle, one can always take $\mathbf{R} \to \mathbf{R} + \mathbf{R}_{\mathbf{init}} = (ct,\mathbf{r}) + (ct_{init},\mathbf{r}_{init})$ with $\mathbf{R}_{\mathbf{init}}$ = a constant 4-Vector due to Poincaré Invariance

General Motion: 10 independent variables = 10 DoF's

4-Position $\mathbf{R} = R^\mu = (ct,\mathbf{r})$ $= \mathbf{R}$

4-Velocity $\mathbf{U} = U^\mu = \gamma(c,\mathbf{u})$ $= d\mathbf{R}/d\tau$

4-Acceleration $\mathbf{A} = A^\mu = \gamma(c\gamma',\gamma'\mathbf{u}+\gamma\mathbf{a})$ $= d\mathbf{U}/d\tau$

4-Jerk $\mathbf{J} = J^\mu = \gamma(c(\gamma\gamma')',(\gamma\gamma'\mathbf{u}+\gamma^2\mathbf{a})')$ $= d\mathbf{A}/d\tau$

$\mathbf{J} = J^\mu = \gamma(c(\gamma'^2+\gamma\gamma''),(\gamma'^2+\gamma\gamma'')\mathbf{u}+\gamma(3\gamma'\mathbf{a}+\gamma\mathbf{j}))$

All Lorentz Scalar Products are Invariants

$(\mathbf{R}\cdot\mathbf{R}) = (ct)^2 - \mathbf{r}\cdot\mathbf{r} = (ct_o)^2 = (c\tau)^2 = -(\mathbf{r_o}\cdot\mathbf{r_o})$:either(±),variable

$(\mathbf{U}\cdot\mathbf{U}) = (c)^2$:temporal(+), fundamental constant

$(\mathbf{A}\cdot\mathbf{A}) = -(a_o)^2 = -(\alpha)^2 = (i\alpha)^2$:spatial(-), variable

$(\mathbf{J}\cdot\mathbf{J}) = (c\gamma_o'')^2-(j_o)^2$:either(±),variable

There are 10 DoF's: {3 [**acceleration**] & 3 [**velocity**] & 4 [**initial 4-Position**]} matches 10 Poincaré symmetries:conservation laws

$(V^{\mu'} = \Lambda^{\mu'}{}_\nu V^\nu + \Delta V[\Delta X^{\mu'}])$ 10 = 6 Lorentz + 4 Translations

General Motion: (alt form **A**,**J**) using $\gamma' = \gamma^3\boldsymbol{\beta}'\cdot\boldsymbol{\beta} = \gamma^3(\mathbf{a}\cdot\mathbf{u})/c^2$

4-Acceleration $\mathbf{A} = A^\mu = (\gamma^4(\mathbf{a}\cdot\mathbf{u})/c,\gamma^4(\mathbf{a}\cdot\mathbf{u})\mathbf{u}/c^2+\gamma^2\mathbf{a})$

4-Jerk $\mathbf{J} = J^\mu = \gamma(c(\gamma^6(\mathbf{a}\cdot\mathbf{u})^2/c^4 + \gamma^4[3\gamma^2(\mathbf{a}\cdot\mathbf{u})^2+(\mathbf{a}\cdot\mathbf{u})']/c^2),(\gamma^6(\mathbf{a}\cdot\mathbf{u})^2/c^4 + \gamma^4[3\gamma^2(\mathbf{a}\cdot\mathbf{u})^2+(\mathbf{a}\cdot\mathbf{u})']/c^2)\mathbf{u}+\gamma(3\gamma^3(\mathbf{a}\cdot\mathbf{u})\mathbf{a}/c^2+\gamma\mathbf{j}))$

Imposed condition: Motion w/ Spatial Orthogonality $(\mathbf{a}\cdot\mathbf{u} = 0) \leftrightarrow (\mathbf{a}\perp\mathbf{u})$

4-Acceleration $\mathbf{A} = A^\mu = \gamma^2(0,\mathbf{a})_\perp$ if $(\mathbf{a}\cdot\mathbf{u})=0$ This also gives $\gamma' = \gamma^3(\mathbf{a}\cdot\mathbf{u})/c^2 \to 0$ which gives γ = constant

4-Jerk $\mathbf{J} = J^\mu = \gamma^3(0,\mathbf{j})_\perp$ if $(\mathbf{a}\cdot\mathbf{u})=0$

Circular Motion ⦚ : constants{ $|\mathbf{r}|$, $|\mathbf{u}|$, $|\mathbf{a}|$, $|\mathbf{j}|$ } ↔ { R , Ω , γ } w/ R=Radius, Ω=AngularFrequency, γ=Relativistic Gamma Factor = Constant 3-vector-magnitudes Motion, but known as constant 3-acceleration-magnitude $|\mathbf{a}|$ = 3D Circular = 4D SR TimeHelix

4-Position $\mathbf{R} = R^\mu = (ct=c\gamma\tau,\mathbf{r} = R\hat{\mathbf{r}} = R(\cos[\Omega t+\theta_o]\hat{\mathbf{n}}_1+\sin[\Omega t+\theta_o]\hat{\mathbf{n}}_2)) = \mathbf{R} = d^0\mathbf{R}/d\tau^0$ $\mathbf{R}\cdot\mathbf{R} = (ct^2) - \mathbf{r}\cdot\mathbf{r} = (ct^2)-R^2$

4-Velocity $\mathbf{U} = U^\mu = \gamma^1(c, \mathbf{u} = R\Omega\hat{\boldsymbol{\theta}} = R\Omega(-\sin[\Omega t+\theta_o]\hat{\mathbf{n}}_1+\cos[\Omega t+\theta_o]\hat{\mathbf{n}}_2)) = d\mathbf{R}/d\tau = d^1\mathbf{R}/d\tau^1$ $\mathbf{U}\cdot\mathbf{U} = \gamma^2(c^2 - \mathbf{u}\cdot\mathbf{u}) = (c)^2$

4-Acceleration $\mathbf{A} = A^\mu = \gamma^2(0, \mathbf{a} = -R\Omega^2\hat{\mathbf{r}} = R\Omega^2(-\cos[\Omega t+\theta_o]\hat{\mathbf{n}}_1-\sin[\Omega t+\theta_o]\hat{\mathbf{n}}_2)) = d\mathbf{U}/d\tau = d^2\mathbf{R}/d\tau^2$ $\mathbf{A}\cdot\mathbf{A} = \gamma^4(0^2 - \mathbf{a}\cdot\mathbf{a}) = -\gamma^4a^2 = -(\alpha)^2$

4-Jerk $\mathbf{J} = J^\mu = \gamma^3(0, \mathbf{j} = -R\Omega^3\hat{\boldsymbol{\theta}} = R\Omega^3(\sin[\Omega t+\theta_o]\hat{\mathbf{n}}_1-\cos[\Omega t+\theta_o]\hat{\mathbf{n}}_2)) = d\mathbf{A}/d\tau = d^3\mathbf{R}/d\tau^3$ $\mathbf{J}\cdot\mathbf{J} = \gamma^6(0^2 - \mathbf{j}\cdot\mathbf{j}) = (c\gamma_o'')^2-(j_o)^2$

○ ↑ ⦚

Circular Motion follows path of ongoing Lorentz Transform $\boldsymbol{\Lambda}\to\mathbf{R}$: (R)otation = Spatial Path+Const Temporal Motion = SR Helix

{ $|\mathbf{r}|$=R , $|\mathbf{u}|$=RΩ , $|\mathbf{a}|$=RΩ^2 , $|\mathbf{j}|$=RΩ^3 } are constants, $(\mathbf{a}\cdot\mathbf{u}) = 0$, $\gamma' = 0$, $\mathbf{a} = (-\Omega^2)\mathbf{r}$, $\mathbf{j} = (-\Omega^2)\mathbf{u}$, $d/d\tau = \gamma d/dt$, $\hat{\mathbf{n}}_1\cdot\hat{\mathbf{n}}_2 = 0$

There are 9 of 10 DoF's: ={1 [initial angle] & 1 [radius] & 3 [**acceleration**] & 4 [**initial 4-Position=location in SpaceTime**]}

={ 1 θ_o , 1 R , 1 Ω , 2 $\hat{\mathbf{n}}_3 = \hat{\mathbf{n}}_1\times\hat{\mathbf{n}}_2$, 4 $(ct_{init},\mathbf{r}_{init})$}

The 1 constraint is: $\mathbf{a} = -(k^2)\mathbf{r} = \mathbf{r}'' = \ddot{\mathbf{r}}$, which gives { $\mathbf{r}$, $\mathbf{u}$, $\mathbf{a}$, $\mathbf{j}$ } ~ ($C^i * \cos[kt] + S^i * \sin[kt]$) for each 3-vector

Boundary Conditions give k=Ω , $\mathbf{a} = -(\Omega^2)\mathbf{r}$, $\mathbf{j} = -(\Omega^2)\mathbf{u}$, $\mathbf{r} = R\hat{\mathbf{r}}$, $\mathbf{u} = R\Omega\hat{\boldsymbol{\theta}}$, $\mathbf{a} = -R\Omega^2\hat{\mathbf{r}}$, $(\mathbf{a}\cdot\mathbf{u}) = 0$, the (cos : sin) mathematics

$|\mathbf{a}| = |\mathbf{u}|^2/|\mathbf{r}| = (R\Omega^2) = (R\Omega)^2/(R)$ or $-\mathbf{a}\cdot\mathbf{r} = (R\Omega)^2 = \mathbf{u}\cdot\mathbf{u}$ [9 DoF's] + [1 constraint] = 10

Hyperbolic Motion)×(: constants { $|\mathbf{R}|$, $|\mathbf{U}|$, $|\mathbf{A}|$, $|\mathbf{J}|$ } ↔ { D , c , α } w/ D=c^2/α=Rindler "Distance", c=LightSpeed, α=ProperAccel = Constant 4-Vector-Magnitudes Motion, but known as constant 4-Acceleration-magnitude $|\mathbf{A}|$ = 4D Hyperbolic

4-Position $\mathbf{R} = R^\mu = (c^2/\alpha)(\sinh[\alpha\tau/c+\xi_o],\cosh[\alpha\tau/c+\xi_o]\ \hat{\mathbf{n}}) = \mathbf{R} = d^0\mathbf{R}/d\tau^0$ $\mathbf{R}\cdot\mathbf{R} = (c^2/\alpha)^2(\sinh^2 - \cosh^2\ \hat{\mathbf{n}}\cdot\hat{\mathbf{n}}) = -(c^2/\alpha)^2 = -D^2$

4-Velocity $\mathbf{U} = U^\mu = (c)(\cosh[\alpha\tau/c+\xi_o],\sinh[\alpha\tau/c+\xi_o]\ \hat{\mathbf{n}}) = d\mathbf{R}/d\tau = d^1\mathbf{R}/d\tau^1$ $\mathbf{U}\cdot\mathbf{U} = (c)^2(\cosh^2 - \sinh^2\ \hat{\mathbf{n}}\cdot\hat{\mathbf{n}}) = (c)^2$

4-Acceleration $\mathbf{A} = A^\mu = (\alpha)(\sinh[\alpha\tau/c+\xi_o],\cosh[\alpha\tau/c+\xi_o]\ \hat{\mathbf{n}}) = d\mathbf{U}/d\tau = d^2\mathbf{R}/d\tau^2$ $\mathbf{A}\cdot\mathbf{A} = (\alpha)^2(\sinh^2 - \cosh^2\ \hat{\mathbf{n}}\cdot\hat{\mathbf{n}}) = -(\alpha)^2$

4-Jerk $\mathbf{J} = J^\mu = (\alpha^2/c)(\cosh[\alpha\tau/c+\xi_o],\sinh[\alpha\tau/c+\xi_o]\ \hat{\mathbf{n}}) = d\mathbf{A}/d\tau = d^3\mathbf{R}/d\tau^3$ $\mathbf{J}\cdot\mathbf{J} = (\alpha^2/c)^2(\cosh^2 - \sinh^2\ \hat{\mathbf{n}}\cdot\hat{\mathbf{n}}) = (\alpha^2/c)^2$

↑ ↔)×(

Hyperbolic Motion follows path of ongoing Lorentz Transform $\boldsymbol{\Lambda}\to\mathbf{B}$: (B)oost = Time·Space Path = SR Hyperbolic

{ $|\mathbf{R}|$=D=c^2/α , $|\mathbf{U}|$=c , $|\mathbf{A}|$=α , $|\mathbf{J}|$=$(\alpha^2/c)^2$ } are constants, $(\mathbf{A}\cdot\mathbf{U})=0$, $\mathbf{A}=(\alpha^2/c^2)\mathbf{R}$, $\mathbf{J}=(\alpha^2/c^2)\mathbf{U}$, γ=cosh : γβ=sinh, τ'=dτ/dt=1/γ : τ=t/γ

There are 8 of 10 DoF's: ={1 [initial hyperangle] & 3 [**proper acceleration**] & 4 [**initial 4-Position=location in SpaceTime**]}

={ 1 ξ_o , 1 α , 2 $\hat{\mathbf{n}}$, 4 $(ct_{init},\mathbf{r}_{init})$}

The constraint eqn. is: $\mathbf{A}=(k^2)\mathbf{R} = \mathbf{R}'' = d^2\mathbf{R}/d\tau^2$, which gives { $\mathbf{R}$, $\mathbf{U}$, $\mathbf{A}$, $\mathbf{J}$ } ~ ($C^\mu * \cosh[k\tau] + S^\mu * \sinh[kt]$) for each 4-Vector

Boundary Conditions give k=(α/c) , $\mathbf{A}=(\alpha^2/c^2)\mathbf{R}$, $\mathbf{J}=(\alpha^2/c^2)\mathbf{U}$, $\mathbf{U}=(c)(\cosh[\alpha\tau/c],\sinh[\alpha\tau/c]\ \hat{\mathbf{n}})$, the (cosh : sinh) mathematics

The 2 constraints: $\mathbf{A}=(\alpha^2/c^2)\mathbf{R}$ splits into (temporal $a^0=(\alpha^2/c^2)r^0$: spatial $\mathbf{a}=(\alpha^2/c^2)\mathbf{r}$)

$|\mathbf{A}| = |\mathbf{U}|^2/|\mathbf{R}| = (\alpha) = (c)^2/(c^2/\alpha) = (c)^2/(D)$ or $-\mathbf{A}\cdot\mathbf{R} = (c)^2 = \mathbf{U}\cdot\mathbf{U}$ [8 DoF's] + [2 constraints] = 10

Relativistic Motion: {4D General, 3D Circular=4D TimeHelix, 4D Hyperbolic, 4D Linear}

SR 4D Linear Motion ×↗ : constants { $\mathbf{a} = \mathbf{0}$ } with $\mathbf{a}$=3-acceleration
= No Forces Minkowski Metric Motion = 4D Linear

4-Position $\mathbf{R} = R^\mu = \gamma\tau(c,\mathbf{u}) = (ct,\mathbf{r})$ $= \mathbf{R}$ $= d^0\mathbf{R}/d\tau^0$ $\mathbf{R\cdot R} = (ct^2) - \mathbf{r\cdot r} = (c\tau)^2$
4-Velocity $\mathbf{U} = U^\mu = \gamma(c,\mathbf{u})$ $= d\mathbf{R}/d\tau = d^1\mathbf{R}/d\tau^1$ $\mathbf{U\cdot U} = \gamma^2(c^2 - \mathbf{u\cdot u}) = (c)^2$
4-Acceleration $\mathbf{A} = A^\mu = (0,\mathbf{a} = \mathbf{0})$ $= d\mathbf{U}/d\tau = d^2\mathbf{R}/d\tau^2$ $\mathbf{A\cdot A} = (0^2 - \mathbf{a\cdot a}) = -(\alpha)^2 = 0$
4-Jerk $\mathbf{J} = J^\mu = (0,\mathbf{j} = \mathbf{0})$ $= d\mathbf{A}/d\tau = d^3\mathbf{R}/d\tau^3$ $\mathbf{J\cdot J} = (0^2 - \mathbf{j\cdot j}) = (c\gamma_o'')^2 - (j_o)^2 = 0$

↑ ↔ ×↗
Linear Motion follows path of ongoing Lorentz Transform $\Lambda \rightarrow \mathbf{I}_{[4]}$: (I)dentity = Time·Space Path = SR Linear
$(\mathbf{a\cdot u}) = 0$, $\gamma' = 0$, $\gamma'' = 0$, γ = constant, $\mathbf{j} = \mathbf{0}$, $\mathbf{R} = \int \mathbf{U}d\tau \rightarrow \mathbf{U}\tau$ for no acceleration = inertial motion
There are 7 of 10 DoF's: ={3 **[initial velocity]** & 4 **[initial 4-Position=location in SpaceTime]**}
={ 3 $\mathbf{u}_{init}$, 4 $(ct_{init},\mathbf{r}_{init})$}
The 3 constraints are: $\mathbf{a} = \mathbf{0}$, which splits into 3 separate components: $a_x = 0$, $a_y = 0$, $a_z = 0$
[7 DoF's] + [3 constraints] = 10

To Reiterate, for 4D SR Motion [# DoF's] + [# Constraints] = 10 due to Poincaré Group:
General: [10 DoF's] + [0 Constraints] = 10
Circular=4DHelix: ⦚ [9 DoF's] + [1 Constraint] = 10 $\mathbf{A} = \gamma^2(0,\mathbf{a} = -R\Omega^2\hat{\mathbf{r}}) \leftrightarrow \mathbf{a} = -(\Omega^2)\mathbf{r}$
Hyperbolic:)×([8 DoF's] + [2 Constraints] = 10 $\mathbf{A}=(\alpha^2/c^2)\mathbf{R}$ splits into (temporal $a^0=(\alpha^2/c^2)r^0$: spatial $\mathbf{a}=(\alpha^2/c^2)\mathbf{r}$
Linear: ×↗ [7 DoF's] + [3 Constraints] = 10 $\mathbf{A} = (0,\mathbf{0}) \leftrightarrow \mathbf{a} = \mathbf{0}$, which splits into $a_x = 0$, $a_y = 0$, $a_z = 0$

General SR Equations:
$\mathbf{R}^{n\prime} = d\mathbf{R}^{(n-1)\prime}/d\tau = d^{(n-1)}\mathbf{R}/d\tau^{(n-1)}$ $\mathbf{J} = \mathbf{A}' = \mathbf{U}'' = \mathbf{R}'''$ $' = d/d\tau = \gamma d/dt$
$\mathbf{r}^{n\prime} = d\mathbf{r}^{(n-1)\prime}/dt = d^{(n-1)}\mathbf{r}/dt^{(n-1)}$ $\mathbf{j} = \mathbf{a}' = \mathbf{u}'' = \mathbf{r}'''$ $' = d/dt$
$(\mathbf{U\cdot U}) = c^2$ is temporal, invariant, fundamental constant
$d/d\tau[\mathbf{U\cdot U}] = d/d\tau[c^2] = 0 = d/d\tau[\mathbf{U}]\cdot\mathbf{U} + \mathbf{U}\cdot d/d\tau[\mathbf{U}] = 2(\mathbf{A\cdot U}) = 0$
$(\mathbf{A\cdot U} = 0) \leftrightarrow (\mathbf{A} \perp \mathbf{U})$: 4-Acceleration (normal to worldline) is orthogonal($\perp$) to 4-Velocity (tangent to worldline)
$d/d\tau[\mathbf{A\cdot U}] = 0 = d/d\tau[\mathbf{A}]\cdot\mathbf{U} + \mathbf{A}\cdot d/d\tau[\mathbf{U}] = \mathbf{J\cdot U} + \mathbf{A\cdot A} = \mathbf{J\cdot U} + -(\alpha)^2$
$(\mathbf{J\cdot U}) = (\alpha)^2 = -(\mathbf{A\cdot A})$
gives $|\mathbf{J}| = |\mathbf{A}|^2/|\mathbf{U}| = (\alpha^2/c) = (\alpha)^2/(c)$

Notice, exact same arguments, one level up, using 4-Vectors:
If SR Hyperbolic Motion:
$(\mathbf{R\cdot R})$ = invariant constant = $-D^2$
$d/d\tau[\mathbf{R\cdot R}] = d/d\tau[\text{constant}] = 0 = d/d\tau[\mathbf{R}]\cdot\mathbf{R} + \mathbf{R}\cdot d/d\tau[\mathbf{R}] = 2(\mathbf{U\cdot R}) = 0$
$(\mathbf{U\cdot R} = 0) \leftrightarrow (\mathbf{U} \perp \mathbf{R})$: 4-Velocity is orthogonal($\perp$) to 4-Position
$d/d\tau[\mathbf{U\cdot R}] = 0 = d/d\tau[\mathbf{U}]\cdot\mathbf{R} + \mathbf{U}\cdot d/d\tau[\mathbf{R}] = \mathbf{A\cdot R} + \mathbf{U\cdot U} = \mathbf{A\cdot R} + (c)^2$
$(\mathbf{A\cdot R}) = -(c)^2 = -(\mathbf{U\cdot U})$
gives $|\mathbf{A}| = |\mathbf{U}|^2/|\mathbf{R}| = (\alpha) = (c)^2/(c^2/\alpha) = (c)^2/(D)$

Notice, exact same arguments, one level up, using 3-vectors:
If SR Circular Motion:
$(\mathbf{r\cdot r})$ = invariant constant = R^2
$d/dt[\mathbf{r\cdot r}] = d/dt[\text{constant}] = 0 = d/dt[\mathbf{r}]\cdot\mathbf{r} + \mathbf{r}\cdot d/dt[\mathbf{r}] = 2(\mathbf{u\cdot r}) = 0$
$(\mathbf{u\cdot r} = 0) \leftrightarrow (\mathbf{u} \perp \mathbf{r})$: 3-velocity is orthogonal($\perp$) to 3-position
$d/dt[\mathbf{u\cdot r}] = 0 = d/dt[\mathbf{u}]\cdot\mathbf{r} + \mathbf{u}\cdot d/dt[\mathbf{r}] = \mathbf{a\cdot r} + \mathbf{u\cdot u} = \mathbf{a\cdot r} + (R\Omega)^2$
$(\mathbf{a\cdot r}) = -(R\Omega)^2 = -(\mathbf{u\cdot u})$
gives $|\mathbf{a}| = |\mathbf{u}|^2/|\mathbf{r}| = (R\Omega^2) = (R\Omega)^2/(R)$

Equivalent forms of 4-Velocity $\mathbf{U} = d\mathbf{R}/d\tau$:
$\mathbf{U} = U^\mu = \gamma(c,\mathbf{u}) = \gamma c(1,\boldsymbol{\beta}) = \gamma c(1,\beta\,\hat{\mathbf{n}}) = c(\gamma,\gamma\beta\,\hat{\mathbf{n}})$
$\mathbf{U} = U^\mu = (c)(\gamma=\cosh[\zeta],\gamma\beta=\sinh[\zeta]\,\hat{\mathbf{n}})$
$\mathbf{U} = U^\mu = (c)(\gamma=[d+1/d]/2,\gamma\beta=[d-1/d]/2\,\hat{\mathbf{n}}) = (c/2)([d+d^{-1}],[d-d^{-1}]\,\hat{\mathbf{n}})$
$= (c/2)([e^\zeta+e^{-\zeta}],[e^\zeta-e^{-\zeta}]\,\hat{\mathbf{n}})$
$= (c)(\cosh[\zeta],\sinh[\zeta]\,\hat{\mathbf{n}})$

$\mathbf{U\cdot U} = \gamma^2(c^2 - \mathbf{u\cdot u}) = (c)^2$
$\mathbf{U\cdot U} = (c)^2(\cosh^2 - \sinh^2\,\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}) = (c)^2$
$\mathbf{U\cdot U} = (c/2)^2([d+1/d]^2 - [d-1/d]^2\,\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}) = (c)^2$
$d = e^\zeta = \gamma(1+\beta) = (1+\beta)/\sqrt{[1-\beta^2]} = \sqrt{[(1+\beta)/(1-\beta)]}$
$(1+\beta)(1-\beta) = 1-\beta^2$
Rapidity $\zeta = \ln[\gamma(1+\beta)]$

4-Position $\mathbf{R} = R^{\mu} = (ct,\mathbf{r})$
4-Acceleration $\mathbf{A} = A^{\mu} = \gamma(c\gamma',\gamma'\mathbf{u}+\gamma\mathbf{a})$

$\gamma' = d\gamma/dt = (\gamma^3\ v\ dv/dt)/c^2 = (\gamma^3\ \mathbf{u}\cdot\mathbf{a})/c^2 = (\mathbf{u}\cdot\mathbf{a_r})/c^2$

$(\mathbf{A}\cdot\mathbf{R}) = (ct,\mathbf{r})\cdot\gamma(c\gamma',\gamma'\mathbf{u}+\gamma\mathbf{a}) = \gamma[\gamma' c^2 t - (\mathbf{r})\cdot(\gamma'\mathbf{u}+\gamma\mathbf{a})] = \gamma[\gamma' c^2 t - (\gamma'\mathbf{r}\cdot\mathbf{u}+\gamma\mathbf{r}\cdot\mathbf{a})] = \gamma[(\gamma^3\ \mathbf{u}\cdot\mathbf{a})(t - \mathbf{r}\cdot\mathbf{u}) - \gamma\mathbf{r}\cdot\mathbf{a})] = -\mathbf{r_o}\cdot\mathbf{a_o}$

<u>If Inertial Motion:</u>
$(\mathbf{R}\cdot\mathbf{R}) = (c\tau)^2 = c^2\tau^2$
$d/d\tau[\mathbf{R}\cdot\mathbf{R}] = d/d\tau[c^2\tau^2] = c^2(d/d\tau)[\tau^2] = 2c^2\tau = d/d\tau[\mathbf{R}\cdot\mathbf{R}] = 2(\mathbf{U}\cdot\mathbf{R})$
$(\mathbf{U}\cdot\mathbf{R}) = (c^2\tau)$
$d/d\tau[\mathbf{U}\cdot\mathbf{R}] = (d/d\tau)(c^2\tau) = (c^2) = d/d\tau[\mathbf{U}\cdot\mathbf{R}] = (\mathbf{A}\cdot\mathbf{R}) + (\mathbf{U}\cdot\mathbf{U}) = (\mathbf{A}\cdot\mathbf{R}) + (c^2)$
$(\mathbf{A}\cdot\mathbf{R}) = 0 = -\mathbf{r_o}\cdot\mathbf{a_o}$
For inertial motion, the proper acceleration $\mathbf{a_o} = \mathbf{0}$

<u>If SR Hyperbolic or Null Motion:</u>
$(\mathbf{R}\cdot\mathbf{R})$ = invariant constant = $-D^2$
$d/d\tau[\mathbf{R}\cdot\mathbf{R}] = d/d\tau[\text{constant}] = 0 = d/d\tau[\mathbf{R}]\cdot\mathbf{R} + \mathbf{R}\cdot d/d\tau[\mathbf{R}] = 2(\mathbf{U}\cdot\mathbf{R}) = 0$
$(\mathbf{U}\cdot\mathbf{R} = 0) \leftrightarrow (\mathbf{U}\perp\mathbf{R})$: 4-Velocity is orthogonal($\perp$) to 4-Position
$d/d\tau[\mathbf{U}\cdot\mathbf{R}] = 0 = d/d\tau[\mathbf{U}]\cdot\mathbf{R}+\mathbf{U}\cdot d/d\tau[\mathbf{R}] = \mathbf{A}\cdot\mathbf{R}+\mathbf{U}\cdot\mathbf{U} = \mathbf{A}\cdot\mathbf{R}+(c)^2$
$(\mathbf{A}\cdot\mathbf{R}) = -(c)^2 = -(\mathbf{U}\cdot\mathbf{U}) = -\mathbf{r_o}\cdot\mathbf{a_o}$

$0 <= \mathbf{r_o}\cdot\mathbf{a_o} <= (c)^2$
$\mathbf{a_o} <= (c)^2/|\mathbf{r_o}| = (c)^2/|c\tau| = c/\tau$

If $\Delta r\ \Delta p >= \hbar/2$, which for light-like $\Delta r\ mc >= \hbar/2$, $\Delta r >= \hbar/2mc$

$\mathbf{a_o} <= (c)^2/|\hbar/2mc| = 2mc^3/\hbar$

$\mathbf{a_o} <= 2mc^3/\hbar$

4-Velocity $\mathbf{U} = U^{\mu} = \gamma(c,\mathbf{u})$ Appendix:
4-Acceleration $\mathbf{A} = A^{\mu} = \gamma(c\gamma',\gamma'\mathbf{u}+\gamma\mathbf{a})$Trying out some math to get a term like "Weak-Gravity" Lorentz Factor $\gamma_{WeakGrav} = 1/\sqrt{[1+2\varphi/c^2-\mathbf{u}\cdot\mathbf{u}/c^2]} = 1/\sqrt{[1-2\mathbf{g}\cdot\mathbf{z}/c^2-\mathbf{u}\cdot\mathbf{u}/c^2]}$, but staying within SR framework:

Action S (relativistic Invariant, Lorentz Scalar Product)
$S = -\int\mathbf{P}\cdot d\mathbf{R} = -\int\mathbf{P}\cdot(d\mathbf{R}/d\tau)d\tau = -\int(\mathbf{P}\cdot\mathbf{U})d\tau = -\int E_o d\tau = -m_o c^2\int d\tau$
$dS = -\mathbf{P}\cdot d\mathbf{R} = -(Edt - \mathbf{p}\cdot d\mathbf{r}) = (-Edt + \mathbf{p}\cdot d\mathbf{r})$
$dS_{temporal\ part} = -Edt$
$dS_{spatial\ part} = \mathbf{p}\cdot d\mathbf{r}$

$S = -\int\mathbf{P}\cdot d\mathbf{R} = -\int\mathbf{P}\cdot(d\mathbf{R}/d\tau)d\tau = -\int(\mathbf{P}\cdot\mathbf{U})d\tau = -\int E_o d\tau = -m_o c^2\int d\tau$, but assumes that $\mathbf{P}$ and thus m_o is constant.

$S = -m_o c^2\int d\tau = -m_o c^2\int(1/\gamma)dt = \int L\, dt$ with $L = -m_o c^2/\gamma$

How about $\mathbf{P}$ variable with just m_o constant?

$S = -\int d(\mathbf{P}\cdot\mathbf{R}) = -\int[d(\mathbf{P})\cdot\mathbf{R} + \mathbf{P}\cdot d(\mathbf{R})] = -\int[d(\mathbf{P})\cdot\mathbf{R} + \mathbf{P}\cdot d(\mathbf{R})](d\tau/d\tau) = -\int[d(\mathbf{P})/d\tau\cdot\mathbf{R} + \mathbf{P}\cdot d(\mathbf{R})/d\tau]d\tau = -\int[d(\mathbf{P})/d\tau\cdot\mathbf{R} + \mathbf{P}\cdot\mathbf{U}]d\tau$
if m_o = constant
$= -m_o\int[d(\mathbf{U})/d\tau\cdot\mathbf{R} + \mathbf{U}\cdot\mathbf{U}]d\tau$
$= -m_o\int[\mathbf{A}\cdot\mathbf{R} + \mathbf{U}\cdot\mathbf{U}]d\tau$
$= -m_o\int[-\mathbf{a_o}\cdot\mathbf{r_o} + c^2]d\tau$

$\Delta\Phi = -(m_o/\hbar)\int[-\mathbf{a_o}\cdot\mathbf{r_o} + c^2]d\tau$

$\Delta\Phi = \int\mathbf{K_{AB}}\cdot d\mathbf{R_{AB}} - \int\mathbf{K_{CD}}\cdot d\mathbf{R_{CD}} = (\omega_{AB}T-\mathbf{k_{AB}}L) - (\omega_{CD}T-\mathbf{k_{CD}}L) = (\mathbf{k_{CD}} - \mathbf{k_{AB}})L$, as temporal components cancel.

$\Delta\Phi = (\mathbf{k_{CD}} - \mathbf{k_{AB}})L = -(m/\hbar)^2\Delta\varphi\lambda L/2\pi = -(m/\hbar)^2 g\Delta z\lambda L/2\pi = -(m/\hbar)^2 gH\sin(\alpha)\lambda L/2\pi = -(m/\hbar)^2 gA_o\sin(\alpha)\lambda/2\pi$

The $\mathbf{A}\cdot\mathbf{R}$ term (acceleration · position) is very similar to the gravitational potential term with $\varphi = -\mathbf{g}\cdot\mathbf{z}$

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Consider a 4D spacetime <event>. It occurs at 4-Position $\mathbf{R} = (ct,\mathbf{r})$

The <event> can be part of a continuous sequence making a worldline. The tangent to that worldline is 4-Velocity $\mathbf{U} = \gamma(c,\mathbf{u})$

If there is a particle [·] (m_o) at the <event>, it can be described as "moving" with 4-Momentum $\mathbf{P} = (E/c,\mathbf{p})$
This motion is related to the tangent via: $\mathbf{P} = m_o\mathbf{U} = (E_o/c^2)\mathbf{U}$

If there is a wave [⸾] (ω_o) at the <event>, it can be described as "moving" with 4-WaveVector $\mathbf{K} = (\omega/c,\mathbf{k})$
This motion is related to the tangent via: $\mathbf{K} = (\omega_o/c^2)\mathbf{U}$

Since both $\mathbf{P}$ and $\mathbf{K}$ are related to $\mathbf{U}$, they must be related to one another.
$\mathbf{P} = (E_o/c^2)\mathbf{U}$
$\mathbf{K} = (\omega_o/c^2)\mathbf{U}$

Thus;
$\mathbf{P} = (E_o/\omega_o)\mathbf{K}$

Since the energy and angular frequency transform the same way:
$\mathbf{P} = (E_o/\omega_o)\mathbf{K} = (\gamma E_o/\gamma\omega_o)\mathbf{K} = (E/\omega)\mathbf{K}$

$\mathbf{P} = (E/\omega)\mathbf{K}$ generally, not just for EM. It holds for all massive:massless particles & waves, generally.

Now, my main contention is that it is an empirical fact, found by classical experiments, that (E/ω) is always found to be ($\hbar$), which appears to be a universal, fundamental, constant. An empirical fact, not requiring a quantum postulate. Doesn't require QM.

4D Stress-Energy Tensors:
The Stress-Energy Tensor $T^{\mu\nu}$ is a general way of defining the physical properties of a relativistic system.
It has dimensional units of [energy_density = pressure]

4-Scalars = 4D (0,0)-Tensors:

Rest Mass	m_o	[kg]	
Rest Energy	$E_o = m_o c^2$	[J = kg·m²/s²]	
Rest Number Density	n_o	[(#/m³)]	
Mass Density	$\rho_{mo} = n_o m_o$	[kg/m³]	
Energy Density	$\rho_{eo} = n_o m_o c^2$	[J/m³ = Pa]	
Pressure	$p_o = p$	[Pa = J/m³]	
Scalar Intensity Factor	Φ	[#J/{rad}²m]	used with Null 4-Vector **K** Null Dust Solution
LightSpeed	c	[m/s]	Universal Constant (speed)
Dirac:Planck Constant	$\hbar = h/2\pi$	[J·s/{rad}]	Universal Constant (action)

4-Vectors = 4D (1,0)-Tensors:

4-Velocity	$\mathbf{U} = U^\mu = \gamma(c,\mathbf{u})$	[m/s]	
4-Momentum	$\mathbf{P} = P^\mu = (E/c,\mathbf{p}) = m_o\mathbf{U}$	[kg·m/s = N·s]	
4-(Dust)NumberFlux	$\mathbf{N} = N^\mu = (nc,\mathbf{n}) = n_o\mathbf{U}$	[#/(m²·s) = (#/m³)·(m/s)]	
4-WaveVector	$\mathbf{K} = K^\mu = (\omega/c,\mathbf{k}) = (\omega_o/c^2)\mathbf{U}$	[{rad}/m]	$(\mathbf{K}\cdot\mathbf{K}) = (\omega/c)^2 - \mathbf{k}\cdot\mathbf{k} = (\omega_o/c)^2 \to 0$ for $\mathbf{K}_{\mathbf{null}}$

4-Tensors = 4D (2,0)-Tensors:

Minkowski 4D Metric	$\eta^{\mu\nu} = V^{\mu\nu} + H^{\mu\nu} \to \mathrm{Diag}[+1,-1,-1,-1]$	[1]	
Temporal Projection	$V^{\mu\nu} = U^\mu U^\nu/c^2 \to \mathrm{Diag}[+1,0,0,0]$	[1]	(V)ertical Projection on LightCone Diagram
Spatial Projection	$H^{\mu\nu} = \eta^{\mu\nu} - V^{\mu\nu} \to \mathrm{Diag}[0,-1,-1,-1]$	[1]	(H)orizontal Projection on LightCone Diagram
Faraday EM	$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \partial \wedge \mathbf{A} = [[0, -e^{0j}/c][+e^{i0}/c, -\varepsilon^{ij}{}_k b^k]]$	[T]	{anti-symmetric}
Stress-Energy	$T^{\mu\nu} \to$ several forms depending on particle types	[Pa=J/m³=N/m²=kg/m·s²]	{symmetric}

$T^{\mu\nu} = [[t^{00}, t^{0j}][t^{i0}, t^{ij}]]$ = [[energy-density,energy-flux/c i-dir],[i^{th} momentum-density*c,i^{th} momentum-flux j-dir]]
$E = mc^2$: energy-flux = momentum-density*c^2 : Since Stress-Energy is symmetric, $t^{0j} = t^{j0}$

Relativistic Fluid Solution: Q^σ = 4-HeatVector, $\Pi^{\mu\nu}$ = ViscousShear
$T^{\mu\nu} = (\rho_{eo})V^{\mu\nu} + (-p_o)H^{\mu\nu} + (\overline{T}^\mu H^\nu{}_\sigma Q^\sigma + Q^\sigma H^\mu{}_\sigma \overline{T}^\nu)/c + \Pi^{\mu\nu} \to [[\rho_e,q^{0j}/c],[q^{i0}/c,p\delta^{ij} + \Pi^{ij}]]$ [N/m² = J/m³ = Pa]

Perfect Fluid Solution:
$T^{\mu\nu} = (\rho_{eo}+p_o)U^\mu U^\nu/c^2 - (p_o)\eta^{\mu\nu} = (\rho_{eo})V^{\mu\nu} + (-p_o)H^{\mu\nu} \to \mathrm{Diag}[\rho_e,p\delta^{ij}]$ [N/m² = J/m³ = Pa]

Matter Dust Solution = Perfect Fluid Solution with no pressure term:
$T^{\mu\nu} = \rho_{mo}U^\mu U^\nu = m_o n_o U^\mu U^\nu = m_o U^\mu n_o U^\nu = P^\mu N^\nu = (\rho_{eo})V^{\mu\nu} \to \mathrm{Diag}[\rho_e,0^{ij}]$ [N/m² = J/m³ = Pa]

Null Dust Solution = Null Fluid Solution = Incoherent Light-Like Radiation:
$T^{\mu\nu} = \Phi K^\mu K^\nu = \Phi(\omega_o/c^2)^2 U^\mu U^\nu$ [(#J/{rad}²m)({rad}/m)({rad}/m) = J/m³] $\mathbf{K}_{\mathbf{null}} = (\omega/c,\mathbf{k}=\omega\hat{\mathbf{n}}/c)$

Maxwell EM Solution = ElectroVacuum Solution
$T^{\mu\nu} = -(1/\mu_o)[F^{\mu\alpha}F^\nu{}_\alpha - (1/4)\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}] \to [[½(\varepsilon_o e^2+b^2/\mu_o), s^{0j}/c],[s^{i0}/c, -\sigma^{ij}]]$

Single Particle Solution:
$T^{\mu\nu} = \int(m_o/\sqrt{[-g]})(dW^\mu/d\tau)(dW^\nu/d\tau)\delta^4[X^\mu-W^\mu]d\tau$ [N/m² = J/m³ = Pa]
with $g = \mathrm{Det}[g^{\mu\nu}]$: $g_{Minkowski} = -1$: $W = W(\tau)$ is WorldLine of Particle
$T^{\mu\nu} = \int(m_o/\sqrt{[-g]})(dW^\mu/d\tau)(dW^\nu/d\tau)\delta[t-t']\delta^3[x^k-w^k](d\tau/dt)\,dt'$
$T^{\mu\nu} = (m_o/\sqrt{[-g]})(dW^\mu/d\tau)(dW^\nu/d\tau)\delta^3[x^k-w^k](d\tau/dt)$
$T^{\mu\nu} = (m_o/\sqrt{[-g]})(dW^\mu/d\tau)(dW^\nu/d\tau)\delta^3[x^k-w^k](d\tau/dt)$
$T^{\mu\nu} = (m_o)(dW^\mu/d\tau)(dW^\nu/d\tau)\delta^3[x^k-w^k](1/\gamma)$

Klein-Gordon Solution (Complex Wave):
$T^{\mu\nu} = (\hbar^2)(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} - \eta^{\mu\nu}\eta^{\alpha\beta})(\partial_\alpha\overline{\psi}\,\partial_\beta\psi) - (\eta^{\mu\nu})(m_o c)^2\overline{\psi}\psi$

Null Dust Solution = Null Fluid Solution = Incoherent Light-Like Radiation:
$T^{\mu\nu} = \Phi K^\mu K^\nu = \Phi(\omega_o/c^2)^2 U^\mu U^\nu$ [$(\#J/\{rad\}^2 m)(\{rad\}/m)(\{rad\}/m) = J/m^3$] $\mathbf{K}_{null} = (\omega/c, \mathbf{k}=\omega\hat{\mathbf{n}}/c)$
If we set factors equal, we find several equivalent forms for the Scalar Intensity Factor Φ of the Null Dust Solution:
$\rho_{mo} U^\mu U^\nu = \Phi(\omega_o/c^2)^2 U^\mu U^\nu$
$\Phi = \rho_{mo}(c^2/\omega_o)^2 = \rho_{eo}(c/\omega_o)^2$
$\Phi = \rho_{eo}(\hbar c/\hbar\omega_o)^2$
$\Phi = n_o E_o(\hbar c/\hbar\omega_o)^2$
$\Phi = n_o E_o(\hbar c/E_o)^2$
$\Phi = n_o(\hbar c)^2/E_o$
$\Phi = n(\hbar c)^2/E$ since $E=\gamma E_o$ and $n=\gamma n_o$
$\Phi = (n/E)(\hbar c)^2$

Taking the temporal-temporal component T^{00} of Maxwell EM and Null Dust
$\frac{1}{2}(\varepsilon_o e^2+b^2/\mu_o) = \rho_{eo} = (n/E)(\hbar c)^2(\omega/c)^2 = (n/E)(\hbar\omega)^2(c/c)^2 = nE = n_o E_o = \rho_{eo}$

The (ħ) factor allows one to use angular frequencies (ω) or energies (E) for the description: $\mathbf{P} = (E/c, \mathbf{p}) = \hbar\mathbf{K} = \hbar(\omega/c, \mathbf{k})$

The Poincaré Group is a Lie Group, and can be written as a Unitary Operation: $U(\Lambda^{\mu'}{}_{\nu}, \Delta X^{\mu'}) = e^{[(i/2\hbar)\omega_{\mu\nu}M^{\mu\nu}]} e^{[(i/\hbar)\Delta X_{\mu}P^{\mu}]}$ with:
4-LinearMomentum P^{μ} [kg·m/s] as generator of SpaceTime-Translation-Transforms with ΔX_{μ} [m] encoding the 1+3=4 displacements
4-AngularMomentum M^{μ} [kg·m²/s] as generator of Lorentz-Transforms with anti-symmetric $\omega_{\mu\nu}$ [1] encoding the 3 angles + 3 boosts
$[\Delta X_{\mu}P^{\mu}]$ and $[\omega_{\mu\nu}M^{\mu\nu}]$ and (ħ) all have dimensional-units of [Action = kg·m²/s = J·s] : Infinitesimal $\Lambda^{\mu'}{}_{\nu} = \delta^{\mu'}{}_{\nu} + \omega^{\mu'}{}_{\nu} + \ldots$

$e^{[(1/2)\omega_{\mu\nu}M^{\mu\nu}]} = e^{(\boldsymbol{\zeta}\cdot\mathbf{K}+\boldsymbol{\theta}\cdot\mathbf{J})}$ with $M^{0j} = -M^{i0} = K^{i}$ and $M^{ij} = \varepsilon^{ij}{}_{k}J^{k}$

Derivatives of Action
from https://en.wikipedia.org/wiki/Conjugate_variables

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The *energy* E of a particle at a certain event is the negative of the derivative of the action along a trajectory of that particle ending at that event with respect to the *time* t of the event.

The *linear momentum* **p** of a particle is the derivative of its action with respect to its *position* **x**.

4-Position $\mathbf{R} = R^{\mu} = (ct,\mathbf{r})$
4-"Position"Gradient $\partial_{\mathbf{R}}{}^{\mu} = \partial/\partial R_{\mu} = (\partial/\partial ct,-\partial/\partial\mathbf{r}) = (\partial_t/c,-\nabla_{\mathbf{r}}) = (\partial_t/c,-\partial_{\mathbf{r}})$

4-Momentum $\mathbf{P} = P^{\mu} = (E/c,\mathbf{p}) = -\partial_{\mathbf{R}}[S_{act}] = -\partial_{R}{}^{\mu}[S_{act}] = -(\partial_t/c,-\nabla_{\mathbf{r}})[S_{act}] = -(\partial_t/c,-\partial_{\mathbf{r}})[S_{act}] = (-\partial_t/c,\partial_{\mathbf{r}})[S_{act}]$
Temporal $E = -\partial_t[S_{act}]$
Spatial $\mathbf{p} = \nabla_{\mathbf{r}}[S_{act}]$

$\mathbf{P} = (m_o)\mathbf{U} = (E_o/c^2)\mathbf{U}$
$\Delta\mathbf{p}\cdot\Delta\mathbf{x} \geq \hbar/2$ and $\Delta E\Delta t \geq \hbar/2$

$\mathbf{P} = -\partial_{\mathbf{R}}[S_{act}] = \partial_{\mathbf{R}}[-S_{act}] = i\hbar\partial_{\mathbf{R}}$
$\mathbf{P}\ f = \partial_{\mathbf{R}}[-S_{act}]\ f = i\hbar\partial_{\mathbf{R}}\ f$
$\partial_{\mathbf{R}}[-S_{act}]\ f = i\hbar\partial_{\mathbf{R}}\ f$
let $f = e^{(g)}$
$\partial_{\mathbf{R}}[-S_{act}]\ f = i\hbar\ f\ \partial_{\mathbf{R}}\ g$
$\partial_{\mathbf{R}}[-S_{act}] = i\hbar\ \partial_{\mathbf{R}}\ g$
$-S_{act} = e^{(i\hbar g)}$
$S_{act} = -e^{(i\hbar g)}$
==
The *electric potential* (φ, voltage) at an event is the negative of the derivative of the action of the electromagnetic field with respect to the density of (free) *electric charge* ρ at that event.

The *magnetic potential* (**a**) at an event is the derivative of the action of the electromagnetic field with respect to the density of (free) *electric current* **j** at that event.

Lagrangian density *L*
$= -1/(4\mu_o)F^{\alpha\beta}F_{\alpha\beta} - A_{\alpha}J_{free}{}^{\alpha} + (1/2)F_{\alpha\beta}M^{\alpha\beta}$
$= (1/2)[\varepsilon_o e^2 - b^2/\mu_o] - \varphi\,\rho_{free} + \mathbf{a}\cdot\mathbf{j}_{\mathbf{free}} + \mathbf{e}\cdot\mathbf{p} + \mathbf{b}\cdot\mathbf{m}$

4-CurrentDensity=4-ChargeFlux $\mathbf{J} = J^{\mu} = (\rho c,\mathbf{j})$
4-"CurrentDensity"Gradient $\partial_{\mathbf{J}}{}^{\mu} = \partial/\partial J_{\mu} = (\partial/\partial\rho c,-\partial/\partial\mathbf{j})$

4-(EM)VectorPotential $\mathbf{A} = A^{\mu} = (\varphi/c,\mathbf{a}) = -\partial_{\mathbf{J}}[S_{EM}] = -\partial_{\mathbf{J}}{}^{\mu}[S_{EM}] = -(\partial_{\rho}/c,-\nabla_{j})[S_{EM}] = -(\partial_{\rho}/c,-\partial_{j})[S_{EM}] = (-\partial_{\rho}/c,\partial_{j})[S_{EM}]$
Temporal $\varphi = -\partial_{\rho}[S_{EM}] = \gamma\varphi_o$
Spatial $\mathbf{a} = \nabla_{j}[S_{EM}] = (\varphi\mathbf{u}/c^2)$

$\mathbf{A} = (\varphi_o/c^2)\mathbf{U}$

==
The *angular momentum* ($\mathbf{l} = \mathbf{r}\wedge\mathbf{p}$) of a particle is the derivative of its action with respect to its *orientation* (angular position).

The *mass-moment* ($\mathbf{n} = t\,\mathbf{p} - m\,\mathbf{r}$) of a particle is the negative of the derivative of its action with respect to its *rapidity*.

4-AngularMomentum $M^{\mu\nu} = [[0,n^{0j}],[n^{i0},\varepsilon^{ij}{}_{k}l^{k}]] = [[0,c\mathbf{n}],[c\mathbf{n}^{T},\mathbf{r}\wedge\mathbf{p}]] = \partial_{\boldsymbol{\theta}}[S^{\nu}] = \partial_{\theta}{}^{\mu}[S^{\nu}]$

4-OrientationTensor $\theta^{\mu\nu} = [[0,\zeta^{0j}],[\zeta^{i0},\varepsilon^{ij}{}_{k}\theta^{k}]]$
$\theta^{0j} = -\theta^{j0} = \zeta^{j} = \boldsymbol{\zeta} = \zeta\hat{\mathbf{n}} = \hat{\mathbf{n}}$ tanh-1 β = rapidity vector
$\theta^{ij} = \varepsilon^{ij}{}_{k}\theta^{k}$ with $\theta^{k} = \theta\hat{\mathbf{n}}$ = angle-axis vector

4-AngularPositionTensor, 4-AngularDisplacementTensor

4-AngularVelocityTensor = $\omega^{\mu\nu} = (d/d\tau)\theta^{\mu\nu}$

Mixed **l**
Spatial **n**

$\Delta\mathbf{l} \cdot \Delta\theta \geq \hbar/2$ and $\Delta\mathbf{l}_x \cdot \Delta\mathbf{l}_y \geq \hbar/2\ \mathrm{Sqrt}[\hbar^2\ |\langle\ \mathbf{l}_z\ \rangle|^2]$ from $\Delta\mathbf{p} \cdot \Delta\mathbf{x} \geq \hbar/2$: $l = mvr = pr$: $\theta = x/r$

$\langle p|x\rangle = (1/\sqrt{[2\pi]})e\text{^}(-ipx)$
$\langle l|\theta\rangle = (1/\sqrt{[2\pi]})e\text{^}(-il\theta)$
==

The *electric field* (**e**) at an event is the derivative of the action of the electromagnetic field with respect to the *electric polarization density* at that event.
The *magnetic induction* (**b**) at an event is the derivative of the action of the electromagnetic field with respect to the *magnetization* at that event.

The Newtonian *gravitational potential* at an event is the negative of the derivative of the action of the Newtonian gravitation field with respect to the *mass density* at that event.
$S = \int L\ dt = \int(L_o/\gamma)(\gamma d\tau) = \int(L_o)(d\tau) = \int L_o d\tau = \int(n_o/n_o)L_o d\tau = \int(n_o L_o)(d\tau/n_o) =?= \int \mathcal{L}\ d^4x$

Lagrangian $L = -m_o c^2/\gamma$
Rest Lagrangian $L_o = -m_o c^2$
Lagrangian density $\mathcal{L} = -n_o m_o c^2 = -\rho_o c^2 = -\mathbf{G}\cdot\mathbf{U}$ with $\mathbf{G} = \rho_o\mathbf{U} = n_o m_o\mathbf{U}$

Lagrangian density $\mathcal{L}$
$= -1/(4\mu_o)F^{\alpha\beta}\ F_{\alpha\beta} - A_\alpha J_{free}{}^{\alpha} + (1/2)F_{\alpha\beta}\ M^{\alpha\beta}$
$= (1/2)[\varepsilon_o e^2 - b^2/\mu_o] - \varphi\ \rho_{free} + \mathbf{a}\cdot\mathbf{j}_{\mathbf{free}} + \mathbf{e}\cdot\mathbf{p} + \mathbf{b}\cdot\mathbf{m}$

Lagrangian density $\mathcal{L}$
$= -(1/8\pi G)(\nabla\Phi)^2 - \rho\Phi$

In Hamiltonian fluid mechanics and quantum hydrodynamics, the *action* itself (or *velocity potential*) is the conjugate variable of the *density* (or *probability density).*

Φ is known as a **velocity potential** for flow velocity **u**. $\mathbf{u} = \nabla[\Phi]$

Application of Noether's theorem allows physicists to gain powerful insights into any general theory in physics, by just analyzing the various transformations that would make the form of the laws involved invariant. For example:

- Invariance of an isolated system with respect to spatial translation (in other words, that the laws of physics are the same at all locations in space) gives the law of conservation of linear momentum (which states that the total linear momentum of an isolated system is constant)
- Invariance of an isolated system with respect to time translation (i.e. that the laws of physics are the same at all points in time) gives the law of conservation of energy (which states that the total energy of an isolated system is constant)

- Invariance of an isolated system with respect to rotation (i.e., that the laws of physics are the same with respect to all angular orientations in space) gives the law of conservation of angular momentum (which states that the total angular momentum of an isolated system is constant)
- Invariance of an isolated system with respect to Lorentz boosts (i.e., that the laws of physics are the same with respect to all inertial reference frames) gives the center-of-mass theorem (which states that the center-of-mass of an isolated system moves at a constant velocity).

TABLE V. Summary of the Lorentz transformations of the scalar and three-dimensional vector quantities of the present work from the G frame to the G′ frame. The G′ frame is moving in the G frame at a constant relative velocity **v**. The number current **Γ** corresponding to the number density n moving with velocity **u** is defined as **Γ** = n**u**. Thus, the Lorentz transformation of the velocity **u** can be obtained by dividing the Lorentz transformation of **Γ** side by side by the Lorentz transformation of n, resulting in **u**′ = −(**v**-**u**), where - denotes the relativistic velocity subtraction, given in the caption of Table I. The velocities $\mathbf{v_a}$, $\mathbf{v_{a0}}$, $\mathbf{v_p}$, and $\mathbf{v_g}$ satisfy this transformation. The quantities (φ/c, **f**), (ct, **r**), (ω/c, **k**), (nc, **Γ**), and (E/c, **p**) are four-vectors. The number currents corresponding to the number densities n_a, n_{a0}, and $n_{a,min}$ of the present work are $\mathbf{\Gamma_a} = n_a\mathbf{v_a}$, $\mathbf{\Gamma_{a0}} = n_{a0}\mathbf{v_{a0}}$, and $\mathbf{\Gamma_{a,min}} = n_{a,min}\mathbf{v_{a0}}$. The transformations of the mass densities $\rho_a = n_aE_a/c^2$, $\rho_{a0} = n_{a0}E_{a0}/c^2$, and $\rho_{a,min} = n_{a,min}E_{a0}/c^2$ follow from the transformations of the number density and single-atom energy [36], using the nonequilibrium and equilibrium single-atom energies $E_a = \gamma_{va}\, m_0c^2$ and $E_{a0} = \gamma_{va0}\, m_0c^2$ and momenta $\mathbf{p_a} = \gamma_{va}\, m_0\mathbf{v_a}$ and $\mathbf{p_{a0}} = \gamma_{va0}\, m_0\mathbf{v_{a0}}$. The subscripts ∥ and ⊥ in this table denote parallel and perpendicular components to the velocity **v**, defined for the generic vector **F** by $\mathbf{F}\| = (\mathbf{F}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}$ and $\mathbf{F}\perp = \mathbf{F} - (\mathbf{F}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}$, where $\hat{\mathbf{v}}$ is the unit vector parallel to **v**.

Faraday EM Tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \partial \wedge \mathbf{A}$ = [[0, $-e^{0j}/c$][$+ei^0/c$, $-\varepsilon^{ij}{}_k b^k$]] Electric field and magnetic flux density

$\mathbf{E}' = \mathbf{E}\| + \gamma_v(\mathbf{E}\perp + \mathbf{v}\times\mathbf{B})$

$\mathbf{B}' = \mathbf{B}\| + \gamma_v(\mathbf{B}\perp - \mathbf{v}\times\mathbf{E} / c^2)$

Electric flux density and magnetic field

$\mathbf{D}' = \mathbf{D}\| + \gamma_v(\mathbf{D}\perp + \mathbf{v}\times\mathbf{H} / c^2)$

$\mathbf{H}' = \mathbf{H}\| + \gamma_v(\mathbf{H}\perp - \mathbf{v}\times\mathbf{D})$

4-AngularMomentum Tensor $M^{\mu\nu} = R^\mu P^\nu - R^\nu P^\mu = \mathbf{R} \wedge \mathbf{P}$ = [[0,n^{0j}],[n^{i0},$\varepsilon^{ij}{}_k l^k$]] = [[0,c**n**],[c$\mathbf{n}^T$,**r**^**p**]] = Angular and boost momenta

$\mathbf{J}' = \mathbf{J}\| + \gamma_v(\mathbf{J}\perp + \mathbf{v}\times\mathbf{N})$

$\mathbf{N}' = \mathbf{N}\| + \gamma_v(\mathbf{N}\perp - \mathbf{v}\times\mathbf{J} / c^2)$

4-Position (ct, **r**) = Time and position

$t' = \gamma_v(t - \mathbf{v}\cdot\mathbf{r} / c^2)$

$\mathbf{r}' = \mathbf{r}\perp + \gamma_v\,(\mathbf{r}\| - \mathbf{v}t)$

4-NumberFlux (nc, **Γ** = n**u**) = Number density and number current

$n' = \gamma_v(n - \mathbf{v}\cdot\mathbf{\Gamma} / c^2)$

$\mathbf{\Gamma}' = \mathbf{\Gamma}\perp + \gamma_v\,(\mathbf{\Gamma}\| - \mathbf{v}n)$

4-WaveVector (ω/c, **k**) = Angular frequency and wave vector

$\omega' = \gamma_v(\omega - \mathbf{v}\cdot\mathbf{k})$

$\mathbf{k}' = \mathbf{k}\perp + \gamma_v(\mathbf{k}\| - \mathbf{v}\omega / c^2)$

4-Momentum (E/c, **p**) = Energy and linear momentum

$E' = \gamma_v\,(E - \mathbf{v}\cdot\mathbf{p})$

$\mathbf{p}' = \mathbf{p}\perp + \gamma_v(\mathbf{p}\| - \mathbf{v}E / c^2)$

4-ForceDensity (φ/c, **f**) = Power-conversion and force densities

$\varphi' = \gamma_v\,(\varphi - \mathbf{v}\cdot\mathbf{f})$

$\mathbf{f}' = \mathbf{f}\perp + \gamma_v(\mathbf{f}\| - \mathbf{v}\varphi / c^2)$

Minkowski Metric

$\eta_{\mu\nu} = \eta^{\mu\nu} \rightarrow \text{Diagonal}[+1,-1,-1,-1]_{(\text{Cartesian})}$

$\eta^{\mu\nu} = V^{\mu\nu} + H^{\mu\nu}$

+1			
	-1		
		-1	
			-1

General Stress-EnergyDensity Tensor $T^{\mu\nu}$: components $t^{\mu\nu}$ = flux of $(\mu)^{\text{th}}$ 4-Momentum component in the (ν)-direcction

t^{00}	t^{01}	t^{02}	t^{03}
t^{10}	t^{11}	t^{12}	t^{13}
t^{20}	t^{21}	t^{22}	t^{23}
t^{30}	t^{31}	t^{32}	t^{33}

t^{00}	t^{0j}
t^{i0}	t^{ij}

$t^{\mu\nu}$

Perfect Fluid Stress-EnergyDensity Tensor, with RestEnergyDensity (ρ_{eo}) & RestIsotropicPressure (p_o)

$T^{\mu\nu} = (\rho_{eo})V^{\mu\nu} + (-p_o)H^{\mu\nu}$

$= \text{Diagonal}[\rho_e,p,p,p]$

$= (\rho_{eo})V^{\mu\nu} + (-p_o)(\eta^{\mu\nu} - V^{\mu\nu})$

$= (\rho_{eo}+p_o)V^{\mu\nu} + (-p_o)(\eta^{\mu\nu})$

$= (\rho_{eo}+p_o)\overline{T}^{\mu}\overline{T}^{\nu} + (-p_o)(\eta^{\mu\nu})$

$= (\rho_{eo}+p_o)U^{\mu}U^{\nu}/c^2 + (-p_o)(\eta^{\mu\nu})$

ρ_e			
	p		
		p	
			p

Rest Enthalpy $w_o = (\rho_{eo}+p_o)$

$T^{\mu\nu} = (\rho_{eo}+p_o)\overline{T}^{\mu}\overline{T}^{\nu} + (-p_o)(\eta^{\mu\nu})$

$T^{00} = (\rho_{eo}+p_o)\overline{T}^{0}\overline{T}^{0} + (-p_o)(\eta^{00}) = \gamma^2(\rho_{eo}+p_o) + (-p_o)$

$T^{0j} = (\rho_{eo}+p_o)\overline{T}^{0}\overline{T}^{j} + (-p_o)(\eta^{0j}) = \gamma^2(\rho_{eo}+p_o)\beta^{j}$

$T^{ij} = (\rho_{eo}+p_o)\overline{T}^{i}\overline{T}^{j} + (-p_o)(\eta^{ij}) = \gamma^2(\rho_{eo}+p_o)\beta^{I}\beta^{j} + p_o\delta^{ij}$

Relativistic Fluid Equations:

$T^{\mu\nu}{}_{,\mu} = \partial_{\mu}T^{\mu\nu} = (\partial/\partial R^{\mu})T^{\mu\nu} = (\partial T^{\mu\nu}/\partial R^{\mu}) = 0^{\nu}$