Top Ten Lists have been created for a variety of categories. Here is a **Physics Top Ten** List, although perhaps in an unexpected way. Using physics to describe our 4D universe, "Ten" occurs in a "number" of very important contexts. Thus... The Physics Top Ten! :)

There are in 4D-SpaceTime: (10) Symmetries, (10) Invariances, (10) Isometries, (10) Conservation Laws ↔ (10) Noether Symmetries, (10) Independent Relativistic-Particle Parameters (using points ·), (10) Independent Relativistic-Fluid Parameters (using densities ☁), (10) Independent { $(6)=(4^2–4)/2$ AntiSymmetric 4-Tensor *Angular* [↻] components + $(4)=(1+3)$ 4-Vector *Linear* [→] components }, $(10)=(4^2+4)/2$ Symmetric 4-Tensor components, (10) Independent (PPN) Parameterized Post-Newtonian formalism variable parameters describing the increasingly unlikely deviations from GR. All of these concepts are due to our observed 4-Dimensional Universe... These <Time·Space> concepts are intimately related to one another and reveal deep, important facts about how our universe operates.

SpaceTime Symmetry is based on Group Theory. 4D SpaceTime uses the Poincaré Symmetry Group *P*, a Lie Group, which has (10) parameters. It is the also known as the Inhomogeneous Lorentz Group, and is the tensor product of the (Homogeneous) Lorentz Symmetry Group *L* with (6) parameters and the SpaceTime-Translation Symmetry Group *ST* with (4) parameters. $P(10)=L(6)\otimes ST(4)$.
Lorentz Symmetry provides: [↻] SpaceTime Measure Isotropy: Same all directions : any rotation angle θ, any boost hyper-angle φ (✳).
SpaceTime-Translation Symmetry provides: [→] SpaceTime Measure Homogeneity: Same all extent : any 4-Displacement $\Delta\mathbf{X}$ [▦].

Invariance is based on the Poincaré Group linear mapping ($V^{\mu'} = \Lambda^{\mu'}{}_{\nu} V^{\nu} + \Delta V[\Delta X^{\mu'}]$) which preserves 4D Interval-Magnitude: ($V^{\mu'}V_{\mu'} = V^{\nu}V_{\nu}$). Note this is similar to algebraic linear mapping x'=A*x+B[Δx]. Observers in different inertial reference frames will obtain the same magnitude results, although they will usually measure different individual component values. Ex.: measuring the same invariant mass m_o but different energies and momenta ($E/c=mc$,$\mathbf{p}=m\mathbf{u}$). The Lorentz Transform 4D (1,1)-Tensor $\Lambda^{\mu'}{}_{\nu}$ has (6) parameters {3 boost + 3 rotate} and the 4-Displacement SpaceTime Transform 4D (1,0)-Tensor [$\Delta X^{\mu'}$] has (4) parameters {1 time + 3 space}, for a total of (10).

Isometry is the concept that various operations can be done to a system and still get the same measurement result $|\Delta\mathbf{X}|$, the SpaceTime Interval or "4D Distance" between <events>. ($\Delta X^{\mu'}\Delta X_{\mu'} = \Delta X^{\nu}\Delta X_{\nu}$) For example, you can rotate a system (3), translate a system in space (3), translate a system in time (1), or boost a system to a uniform velocity (3), for a total of (10). Isometry = "Same Measure". This is related to the concept of active (change the physical system) transformations and passive (change the coordinate system) transformations.

Conservation Laws are the broad principles that govern physical systems. They describe those properties which remain unchanged in proper time for a closed system. Noether's Theorem: "If a system has a continuous symmetry property, then there are corresponding quantities whose values are conserved in time." The most famous is the Conservation of Energy (1), from time translation, followed by the Conservation of Linear Momentum (3) from space translation, and Conservation of Angular Momentum (3) from spatial rotation. Another, less well-known, is the Conservation of Mass Moment (3) from temporal-spatial Lorentz boosts. Total (10) & (10).

Particle Parameters are those independent variables that fully describe the SpaceTime dynamics of a point particle (·). These include [↻] 4-AngularMomentum $M^{\alpha\beta}$, which has a total of $(6)=(4^2-4)/2$ independent parameters due to being an anti-symmetric 4D (2,0)-Tensor, and [→] 4-LinearMomentum P^{μ}, which has a total of (4) parameters due to being a 4D (1,0)-Tensor=4-Vector. Combined, there are a total of (10) independent parameters. One can also think of this as particle Spin-momentum [↻] + Linear-momentum [→].

Fluid Parameters are those independent variables that fully describe the SpaceTime dynamics of a fluid (☁). The components of the Relativistic Fluid Stress-Energy 4-Tensor $T_{relfluid}{}^{\mu\nu}$ include energy-density (1), heat-flux (3), isotropic pressure (1), and anisotropic viscous-shear (5). Likewise, a symmetric 4D (2,0)-Tensor has a total of $(10)=(4^2+4)/2$ independent parameters. It is technically a Stress-EnergyDensity 4-Tensor, as each component has units of [$kg/(m\cdot s^2) = J/m^3 = N/m^2 = Pa$ = {energy-density} = {pressure} = {stress}], but often called Stress-Energy. In SR, it has the property of being divergenceless: ($\partial_{\mu}T^{\mu}{}_{\nu} = 0_{\nu}$ or $\partial^{\mu}T_{\mu}{}^{\nu} = 0^{\nu}$) which is a set of 4 continuity equations along the 4 dimensions.

4D Tensor Components are those independent components of a generic 4D Tensor. For a regular, generic 2-index tensor there are $N\times N = N^2$ components and for a regular, generic 1-index tensor there are N components. Any 2-index tensor $T^{\mu\nu}$ can be decomposed into symmetric $S^{\mu\nu}$ and anti-symmetric $A^{\mu\nu}$ parts using the following procedure. Define $S^{\mu\nu} = (T^{\mu\nu} + T^{\nu\mu})/2$ and $A^{\mu\nu} = (T^{\mu\nu} - T^{\nu\mu})/2$.
$S^{\mu\nu} + A^{\mu\nu} = (T^{\mu\nu} + T^{\nu\mu})/2 + (T^{\mu\nu} - T^{\nu\mu})/2 = 2T^{\mu\nu}/2 + T^{\nu\mu}/2 - T^{\nu\mu}/2 = T^{\mu\nu}$.
By inspection (just switch indices), the symmetric 4-Tensor $S^{\mu\nu} = +S^{\nu\mu}$ and the anti-symmetric 4-Tensor $A^{\mu\nu} = -A^{\nu\mu}$.
Viewing the tensors as [N×N] matrices:
The symmetric tensor has $(N^2+N)/2$ independent components. The lower triangle = upper triangle. The diagonal has N components.
The anti-symmetric tensor has $(N^2-N)/2$ independent components. The lower triangle = –upper triangle. The diagonal has all zeroes.
So, for 4D tensors we obtain:
The symmetric tensor has $(4^2+4)/2 = (10)$ independent components.
The anti-symmetric tensor has $(4^2-4)/2 = (6)$ independent components, + 4-Vector (4) components = (10) independent components.

Parameterized Post-Newtonian formalism (PPN), is a formulation that explicitly details the parameters in which a general theory of gravity can differ from Newtonian gravity. It is used as a tool to compare Newtonian and Einsteinian gravity in the limit of a weak gravitational field: {$g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}$ with $|h^{\mu\nu}|$ small} and in which objects move slowly compared to the speed of light {$|v|\ll c$}.
In general, PPN formalism can be applied to all (inc. alternative) metric theories of gravitation in which the massive bodies satisfy the Einstein Equivalence Principle (EEP). There are (10) Parameters. To-date, GR is the best fit to measured data and observations.

This wondrous number (10) occurs because of several reasons: Our spacetime has a 4D Poincaré Group Symmetry, and 4D Tensor mathematics provides an excellent mathematical framework to describe the <events> of our universe. The Poincaré Group naturally splits into <Time·Space> components, with each tensor splitting into temporal and/or spatial and/or mixed components. The two Casimir Invariants of the Poincaré Group, those components which commute with all other components of the Poincaré Group, lead naturally to a linear [→] (mass-momentum) & an angular [⭮] (spin-momentum) splitting.
Let's look at some of the physical, geometric, tensorial objects describing these <events>.

4-Vectors = 4D (1,0)-Tensors:

4-Position	$X^\mu = (ct,\mathbf{x}) = \mathbf{X} \in$ <event>	[m]	$(ct,\mathbf{x}) \to (ct,x,y,z)$ only Lorentz Invariant, not Poincaré Invariant Alt. $R^\mu = \mathbf{R}$
4-Displacement	$\Delta X^\mu = (c\Delta t,\Delta\mathbf{x}) = \Delta\mathbf{X}$	[m]	Finite $\Delta\mathbf{X} = \mathbf{X_2} - \mathbf{X_1}$ fully Poincaré Invariant
4-Differential	$dX^\mu = (cdt,d\mathbf{x}) = d\mathbf{X}$	[m]	Infinitesimal $d\mathbf{X}$ fully Poincaré Invariant
4-Gradient	$\partial^\mu = (\partial_t/c,-\nabla) = \boldsymbol{\partial} = \partial/\partial X_\mu$	[1/m]	$X_\mu = \eta_{\mu\nu} X^\nu : X^\mu = \eta^{\mu\nu} X_\nu$: 4D One-form $\partial_\mu = (\partial_t/c,\nabla) = \partial/\partial X^\mu$
4-Velocity	$U^\mu = \gamma(c,\mathbf{u}) = \mathbf{U}$	[m/s]	$= d\mathbf{X}/d\tau$: Lorentz Gamma $\gamma = 1/\sqrt{[1-(u/c)^2]} = dt/d\tau$
4-Acceleration	$A^\mu = \gamma(c\gamma',\gamma'\mathbf{u}+\gamma\mathbf{a}) = \mathbf{A}$	[m/s²]	$= d\mathbf{U}/d\tau = d^2\mathbf{X}/d\tau^2$
4-"Unit"Temporal	$\bar{T}^\mu = \gamma(1,\boldsymbol{\beta}) = \bar{\mathbf{T}} = \mathbf{U}/c$	[1]	Dimensionless primitive 4-Vector: also 4-"Unit"Spatial $\bar{\mathbf{S}}$
4-LinearMomentum	$P^\mu = (E/c=mc,\mathbf{p}=m\mathbf{u}) = \mathbf{P} = m_o\mathbf{U}$	[(kg·m)/s = N·s]	m_o = RestMass : $E = mc^2 = \gamma m_o c^2 = \gamma E_o$
4-TotalMomentum	$P_T^\mu = (H/c,\mathbf{p_T}) = \mathbf{P_T} = \mathbf{P}+q\mathbf{A_{EM}}$	[(kg·m)/s = N·s]	Includes effects of particle in EM potential $\mathbf{Q} = q\mathbf{A_{EM}}$

4-Tensors,Upper = 4D (2,0)-Tensors:

Minkowski 4D Metric	$\eta^{\mu\nu} = \partial^\mu[X^\nu] = V^{\mu\nu} + H^{\mu\nu}$	→ Diag[+1,-1,-1,-1]	[1] Diag[+1,-1,-1,-1] in Cartesian, different in other coords
Kronecker Delta (2,0)	$\delta^{\mu\nu}$	= Diag[+1,+1,+1,+1]	[1] $[\delta^{\mu\nu}]$ = {1 if $^{\mu=\nu}$, 0 otherwise}
Temporal Projection (2,0)	$V^{\mu\nu} = U^\mu U^\nu/c^2 = \bar{T}^\mu\bar{T}^\nu$	→ Diag[+1,0,0,0]	[1] (V)ertical Projection on LightCone ST Diagram
Spatial Projection (2,0)	$H^{\mu\nu} = \eta^{\mu\nu} - V^{\mu\nu}$	→ Diag[0,-1,-1,-1]	[1] (H)orizontal Projection on LightCone ST Diagram
4-AngularTensor	$\omega^{\mu\nu}$		[{rad}] encoding 3 rotation angles + 3 boost angles
4-AngularMomentum	$M^{\alpha\beta} = X^\alpha P^\beta - X^\beta P^\alpha = \mathbf{X}\wedge\mathbf{P}$		[(kg·m²)/s = N·m·s = J·s = Action]
Relativistic Fluid Stress-Energy(Density) 4-Tensors	$T_{RelFluid}^{\mu\nu} : T_{PerfectFluid}^{\mu\nu}$		[kg/(m·s²) = J/m³ = N/m² = Pa]

4-Tensors,Mixed = 4D (1,1)-Tensors:

Lorentz Transform	$\Lambda^{\mu'}{}_\nu = \partial_\nu[X^{\mu'}] = \partial X^{\mu'}/\partial X^\nu$		[1] {Rotation $R^{\mu'}{}_\nu$: Boost $B^{\mu'}{}_\nu$: Flip $F^{\mu'}{}_\nu$: Combo : Identity}
Kronecker Delta (1,1)	$\delta^\mu{}_\nu = \eta^\mu{}_\nu = g^\mu{}_\nu$	= Diag[+1,+1,+1,+1]	[1] $[\delta^\mu{}_\nu]$ = {1 if $^\mu=_\nu$, 0 otherwise} Note the sign flips in $\eta^\mu{}_\nu$
Temporal Projection (1,1)	$V^\mu{}_\nu = U^\mu U_\nu/c^2 = \bar{T}^\mu\bar{T}_\nu$	→ Diag[+1,0,0,0]	[1] (V)ertical Projection on LightCone Diagram
Spatial Projection (1,1)	$H^\mu{}_\nu = \eta^\mu{}_\nu - V^\mu{}_\nu = \eta_{\sigma\nu} H^{\mu\sigma}$	→ Diag[0,+1,+1,+1]	[1] (H)orizontal Projection on LightCone Diagram

4-Tensors,Lower = 4D (0,2)-Tensors:

Kronecker Delta (0,2)	$\delta_{\mu\nu}$	= Diag[+1,+1,+1,+1]	[1] $[\delta_{\mu\nu}]$ = {1 if $_{\mu=\nu}$, 0 otherwise}
Temporal Projection (0,2)	$V_{\mu\nu} = U_\mu U_\nu/c^2 = \bar{T}_\mu\bar{T}_\nu$	→ Diag[+1,0,0,0]	[1] (V)ertical Projection on LightCone Diagram
Spatial Projection (0,2)	$H_{\mu\nu} = \eta_{\mu\nu} - V_{\mu\nu}$	→ Diag[0,-1,-1,-1]	[1] (H)orizontal Projection on LightCone Diagram

4-Tensors can be transformed among {Upper,Mixed,Lower} forms by Tensor Index Raising:Lowering using the Minkowski Metric $\boldsymbol{\eta}$.
Another tensor object which comes up often is the Levi-Civita Totally AntiSymmetric Tensor: 3D $\varepsilon_{ij}{}^k$: 4D $\varepsilon_{\mu\nu\rho}{}^\sigma$: & other combinations.

Integral to physical, relativistic tensors is that {4D <Time·Space> = 1D Temporal (t) + 3D Spatial (x,y,z)} entities have specific ways of splitting into their various natural, measurable components, depending on the type of tensor that they are represented by:

4-Scalar	$S = S$	(1) Invariant Lorentz Scalar, same for all frames {s} or {s_o}	1 {4D (0,0)-Tensor} component
4-Vector	$\mathbf{V} = V^\mu$	($1^{\,0}+3^{\,j}$)-splitting into {v^t,v^x,v^y,v^z}	4 {4D (1,0)-Tensor} components
4-Tensor,AntiSymmetric	$\boldsymbol{T_{asym}} = T_{asym}{}^{\mu\nu}$	($3^{\,0j}+3^{\,j\neq k}$)-splitting into {$t^{tx},t^{ty},t^{tz},t^{xy},t^{xz},t^{yz}$} w/ all $^{j=k}$ comps=0	6 {4D (2,0)-Tensor} components
4-Tensor,Symmetric	$\boldsymbol{T_{sym}} = T_{sym}{}^{\mu\nu}$	($1^{\,00}+3^{\,0j}+3^{\,j=k}+3^{\,j\neq k}$)-splitting into {$t^{tt},t^{tx},t^{ty},t^{tz},t^{xx},t^{yy},t^{zz},t^{xy},t^{xz},t^{yz}$}	10 {4D (2,0)-Tensor} components
4-Tensor,Generic	$\boldsymbol{T} = T^{\mu\nu}$	[[$t^{tt},t^{tx},t^{ty},t^{tz}$],[$t^{xt},t^{xx},t^{xy},t^{xz}$],[$t^{yt},t^{yx},t^{yy},t^{yz}$],[$t^{zt},t^{zx},t^{zy},t^{zz}$]]	16 {4D (2,0)-Tensor} components

There are relativistic Symmetries/Operations in nature which leave the interval-measurement between events unchanged (invariant) and lead to fundamental Conservation Laws. These can use active or passive transformations, including changes of coordinate basis.
SR 4-Vectors have a Poincaré Group linear mapping ($V^{\mu'} = \Lambda^{\mu'}{}_\nu V^\nu + \Delta V[\Delta X^{\mu'}]$) which preserves interval-magnitude: ($V^{\mu'}V_{\mu'} = V^\nu V_\nu$).
The Poincaré Group {Lorentz Group $\Lambda^\mu{}_\nu$ + SpaceTime Translation Group ΔX^μ} is the Full SpaceTime Symmetry Group. $\mathbf{R}^{1,3} \ltimes O(1,3)$
(10) Isometries match the {Anti-Symmetric 4D (2,0)-Tensor [3+3]-splitting→(6) + 4D (1,0)-Tensor (1+3)-splitting→(4)}.
(10) Isometries match the {Symmetric 4D (2,0)-Tensor [1+3+3+3]-splitting→(10)}.

Notation / Conventions / Fundamentals:
Tensor Convention Used: {Temporal, 0^{th} Component, Positive(+), SI} = Metric Signature (+,-,-,-) with [SI Dimensional-Units].
aka. {"Time-Positive", "Particle-Physics", "West-Coast", "Mostly-Minuses"} Metric Sign Convention → The "Metric System" :-)

SR <Time·Space>-splitting Component Coloring Mnemonic: Temporal (blue) + Spatial (red) give Mixed SpaceTime (purple)

4D "Flat" <Time·Space> SR:Minkowski Metric $\eta_{\mu\nu} = \eta^{\mu\nu} \rightarrow \text{Diagonal}[+1,-1,-1,-1]_{(\text{Cartesian})}$: Generally, $\{g_{\mu\nu}\}=1/\{g^{\mu\nu}\}$ for non-zero
Mixed 4D (1,1)-Tensor form Minkowski Metric $\eta^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} = \text{Diagonal}[+1,+1,+1,+1]_{(\text{Always})} = I_{[4]} = g^{\mu}{}_{\nu}$ = Kronecker Delta = Identity_{4D}

4-Position $\mathbf{R} = R^{\mu} = (ct,\mathbf{r}) = \eta^{\mu\nu}R_{\nu}$ alt. $\mathbf{X} = X^{\mu}$: 4D Position-OneForm $R_{\mu} = (ct,-\mathbf{r}) = \eta_{\mu\nu}R^{\nu}$ [m]
4-Gradient $\partial = \partial^{\mu} = (\partial_t/c,-\nabla) = (\partial/\partial R_{\mu}) = \eta^{\mu\nu}\partial_{\nu}$: 4D Gradient-OneForm $\partial_{\mu} = (\partial_t/c,\nabla) = (\partial/\partial R^{\mu}) = \eta_{\mu\nu}\partial^{\nu}$ [1/m]

$\partial^{\mu}[R^{\nu}] = \partial[\mathbf{R}] = (\partial_t/c,-\nabla)[(ct,\mathbf{r})] \rightarrow (\partial_t/c,-\partial_x,-\partial_y,-\partial_z)[(ct,x,y,z)]$ = Jacobian = $\text{Diagonal}[+1,-1,-1,-1]_{(\text{Cartesian})} = \eta^{\mu\nu}$ = SR:Minkowski Metric
$\partial^{\mu}\,\eta_{\mu\nu}\,R^{\nu} = (\partial\cdot\mathbf{R}) = (\partial_t/c,-\nabla)\cdot(ct,\mathbf{r}) \rightarrow (\partial_t/c,-\partial_x,-\partial_y,-\partial_z)\cdot(ct,x,y,z) = (\partial_t ct/c+\partial_x x+\partial_y y+\partial_z z) = 4$ = SR 4D SpaceTime Dimension
$\partial_{\nu}[R^{\mu'}] = \partial R^{\mu'}/\partial R^{\nu} = \Lambda^{\mu'}{}_{\nu}$ = SR Lorentz Transformation: $A^{\mu'} = \Lambda^{\mu'}{}_{\nu}A^{\nu}$ = New.Ref.Frame' = LorentzTransf. contracted w/ Old.Ref.Frame

4D Tensors use Greek indices: ex. { μ, ν, σ, ρ, ...} : ex. 4-Position $R^{\mu} = (r^{\mu}) = (r^0,r^1,r^2,r^3)$, with 4 possible index-values {0,1,2,3}
3D tensors use Latin indices: ex. { i, j, k, …} : ex. 3-position $r^k = (r^k) = (r^1,r^2,r^3)$, with 3 possible index-values {1,2,3}

4-Vector (4D) $\mathbf{A} = A^{\mu} = (a^{\mu}) = (a^0,\mathbf{a}) = (a^0,a^k) = (a^0,a^1,a^2,a^3) \rightarrow (a^t,a^x,a^y,a^z)_{[\text{Cartesian:rectangular}]} \rightarrow (a^t,a^r,a^{\theta},a^{\varphi})_{[\text{spherical}]}$ → other coordinate basis
3-vector (3D) $\mathbf{a} = a^k = (a^k) = (\mathbf{a}) = (a^k) = (a^1,a^2,a^3) \rightarrow (a^x,a^y,a^z)_{[\text{Cartesian:rectangular}]} \rightarrow (a^r,a^{\theta},a^{\varphi})_{[\text{spherical}]}$ → other coordinate basis
4D Dot Product $A^{\mu}\,\eta_{\mu\nu}\,B^{\nu} = (\mathbf{A}\cdot\mathbf{B}) = (a^0,\mathbf{a})\cdot(b^0,\mathbf{b}) = (+a^0b^0-a^1b^1-a^2b^2-a^3b^3) = (a^0b^0-\mathbf{a}\cdot\mathbf{b})$
3D dot product $a^j\,\delta_{jk}\,b^k = (\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a})\cdot(\mathbf{b}) = (+a^1b^1+a^2b^2+a^3b^3)$

4-Scalar	$S = S$	(1) Invariant Lorentz Scalar, same for all frames, {s} or $\{s_o\}$	1	{4D (0,0)-Tensor} component
4-Vector	$\mathbf{V} = V^{\mu}$	$(1\ ^0 + 3\ ^j)$-splitting into $\{v^t,v^x,v^y,v^z\}$	4	{4D (1,0)-Tensor} components
4-Tensor,$_{\text{AntiSymmetric}}$	$\boldsymbol{T_{asym}} = T_{asym}{}^{\mu\nu}$	$(3\ ^{0j} + 3\ ^{j\neq k})$-splitting into $\{t^{tx},t^{ty},t^{tz},t^{xy},t^{xz},t^{yz}\}$ w/ all $^{j=k}$ comps=0	6	{4D (2,0)-Tensor} components
4-Tensor,$_{\text{Symmetric}}$	$\boldsymbol{T_{sym}} = T_{sym}{}^{\mu\nu}$	$(1\ ^{00} + 3\ ^{0j} + 3\ ^{j=k} + 3\ ^{j\neq k})$-splitting into $\{t^{tt},t^{tx},t^{ty},t^{tz},t^{xx},t^{yy},t^{zz},t^{xy},t^{xz},t^{yz}\}$	10	{4D (2,0)-Tensor} components
4-Tensor,$_{\text{Generic}}$	$\boldsymbol{T} = T^{\mu\nu}$	$[[t^{tt},t^{tx},t^{ty},t^{tz}],[t^{xt},t^{xx},t^{xy},t^{xz}],[t^{yt},t^{yx},t^{yy},t^{yz}],[t^{zt},t^{zx},t^{zy},t^{zz}]]$	16	{4D (2,0)-Tensor} components

S = {s} = scalar $(1=4^0)$: $\mathbf{V} = V^{\mu} = (v^0,\mathbf{v}=v^i)$ = vector $(4=4^1)$: $\boldsymbol{T} = \boldsymbol{T_{asym}} + \boldsymbol{T_{sym}} = T^{\mu\nu} = [[t^{00}, t^{0k}], [t^{j0}, t^{jk}]]$ = matrix $(16=4^2)$: etc.
Technically, these all are 4-Tensors = 4D Tensors; specified precisely using the #D (m,n)-Tensor notation {# dims, m $^{\text{upper}}$, n $_{\text{lower}}$ indices}
All SR 4-Tensors obey $T^{\mu 1'\ldots\mu m'} = \Lambda^{\mu 1'}{}_{\nu 1}\,\Lambda^{\mu 2'}{}_{\nu 2} \ldots \Lambda^{\mu m'}{}_{\nu m}\,T^{\nu 1\ldots\nu m}$: m is the # of indices and a separate Lorentz Transform Λ for each index

<Time·Space> 4-Vector Name matches its spatial 3-vector component name: ex. 4-Position $\mathbf{R}$ = (c*time t,3-position $\mathbf{r}$) [length]→[m]
LightSpeed Factor (c) will be in temporal component as required to make all [dimensional-units] of a 4-Vector's components match
SR 4-Vector $\mathbf{V}$ = (**4D SpaceTime 4-Vector**) = (1D temporal 3-scalar, **3D spatial 3-vector**) → 4D (1+3)-splitting into (v^t,v^x,v^y,v^z)

Tensor-index-notation in non-bold: ex. $A^{\mu} = (a^{\mu}) = (a^0,a^j) = (a^0,a^1,a^2,a^3)$: ex. $A^{\mu\nu} = [[a^{\mu\nu}]] = [[a^{00}, a^{0k}],[a^{j0}, a^{jk}]]$
4-Vectors (4D) in bold UPPERCASE: ex. $\mathbf{A} = \bar{\bar{\mathbf{A}}} = (\mathbf{A}) = (a^0,\mathbf{a}) = (a^0,a^1,a^2,a^3)$
3-vectors (3D) in bold lowercase: ex. $\mathbf{a} = \bar{\bar{\mathbf{a}}} = \vec{a} = (\mathbf{a}) = (\mathbf{a}) = (a^1,a^2,a^3)$
Temporal scalars (1D) in non-bold, usually lowercase, 0^{th} component: ex. a^0, a_0 "Count from 1, but index from 0 :-) "
Individual non-grouped components of 4-Tensors in non-bold: ex. $\mathbf{A} = (a^0,a^1,a^2,a^3) = (a^0,\mathbf{a})$ Vectors are grouped structures, thus bold
Rest scalars (invariants) in non-bold, denoted with naught ($_o$): ex. m_o : from $\mathbf{P} = m_o\mathbf{U}$ "A rest-frame is a valid relativistic concept"

Upper index 4-Vector $\mathbf{A} = \bar{A} = A^{\mu} = (a^{\mu}) = (a^0,a^j)$: Lower index 4-CoVector $\underline{B} = B_{\mu} = (b_{\mu}) = (b_0,b_j)$ a.k.a 4-DualVector=4D-OneForm
Index lowering/raising via Minkowski Metric $\boldsymbol{\eta}$: ex. $R_{\mu} = \eta_{\mu\nu}R^{\nu}$ or $\partial^{\mu} = \eta^{\mu\nu}\partial_{\nu}$ or $U^{\mu} = \eta^{\mu\nu}U_{\nu}$ with 4-Velocity $\mathbf{U} = U^{\mu} = \gamma(c,\mathbf{u}) = \gamma c(1,\boldsymbol{\beta})$

SR Relativistic Gamma $\gamma = 1/\sqrt{[1-\boldsymbol{\beta}\cdot\boldsymbol{\beta}]} = dt/d\tau$: Relativistic $\boldsymbol{\beta} = \mathbf{u}/c = \{0..1\}\hat{\mathbf{n}}$: ProperTimeDerivative $(d/d\tau) = \gamma(d/dt) = (\mathbf{U}\cdot\partial)$
Derived from $\mathbf{U} = d\mathbf{R}/d\tau = (d/d\tau)\mathbf{R} = (d/d\tau)(dt/dt)\mathbf{R} = (dt/d\tau)(d/dt)\mathbf{R} = (dt/d\tau)(d/dt)(ct,\mathbf{r}) = (dt/d\tau)(c,\mathbf{u}) = \gamma(c,\mathbf{u}) = \mathbf{U}$

4D (1,0)-Tensor = 4-Vector: $\mathbf{A} = \bar{\bar{\mathbf{A}}} = A^{\mu}$: ex. 4-Momentum $\mathbf{P} = P^{\mu} = (E/c,\mathbf{p}) = (mc,m\mathbf{u}) = m_o\mathbf{U} = m_o\gamma(c,\mathbf{u}) = m(c,\mathbf{u})$
4D (0,1)-Tensor = 4-CoVector = 4D-OneForm: $\underline{\underline{\mathbf{A}}} = A_{\mu}$: ex. 4D GradientOneForm $\partial_{\mu} = (\partial_t/c,\nabla) = (\partial/\partial R^{\mu})$

"Unit"Temporal 4-Vector $\bar{\mathbf{T}} = \gamma(1,\boldsymbol{\beta})$, with Lorentz Scalar Invariant $\bar{\mathbf{T}}\cdot\bar{\mathbf{T}} = T^{\mu}T_{\mu} = \gamma^2[1^2-\boldsymbol{\beta}\cdot\boldsymbol{\beta}] = +1$ $\bar{\mathbf{T}} = \mathbf{U}/c$
Null 4-Vector $\bar{\mathbf{N}} \sim (\pm|a|,\mathbf{a}) = a(\pm 1,\hat{\mathbf{n}})$, with Lorentz Scalar Invariant $\bar{\mathbf{N}}\cdot\bar{\mathbf{N}} = N^{\mu}N_{\mu} = a^2[1^2-\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}] = 0$
"Unit"Spatial 4-Vector $\bar{\mathbf{S}} = \gamma_{\beta\hat{n}}(\boldsymbol{\beta}\cdot\hat{\mathbf{n}},\hat{\mathbf{n}})$, with Lorentz Scalar Invariant $\bar{\mathbf{S}}\cdot\bar{\mathbf{S}} = S^{\mu}S_{\mu} = \gamma_{\beta\hat{n}}{}^2[(\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^2-\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}] = -1$ $\bar{\mathbf{T}}\cdot\bar{\mathbf{S}} = (\gamma^*\gamma_{\beta\hat{n}})[\boldsymbol{\beta}\cdot\hat{\mathbf{n}}-\boldsymbol{\beta}\cdot\hat{\mathbf{n}}] = 0$
$\bar{\mathbf{T}}\cdot\bar{\mathbf{S}} = 0 \leftrightarrow (\bar{T} \perp_{4D} \bar{S})$

Time-like separated <Events>
Invariant Temporal Causality=Time-ordering
Moving Clock = ←|Time Dilation|→
Relativity of Stationarity = non-Topological

Null-like separated <Events>
Invariant Null LightCone
| |Invariant LightSpeed (c)| |
Causal & Topological

Space-like separated <Events>
Invariant Spatial Topology=Space-ordering
Moving Ruler = →|Length Contraction|←
Relativity of Simultaneity = non-Causal

Relativistic Particle · Scalar Cartesian Form: (all scalars): ***fully split into each individual 4D dimension*** {t,x,y,z}

#	Dim-Type	Isometry Operation	Conservation Law	# Parameters	Measure Symmetry
1	Mixed	Lorentz Boost $\Lambda^{\mu'}{}_{\nu} \to B^{\mu'}{}_{\nu}[t,x]$	mass-moment n^{tx}	1 [↻]	Isotropy-t,x : boost along t-x, φ_x
2	Mixed	Lorentz Boost $\Lambda^{\mu'}{}_{\nu} \to B^{\mu'}{}_{\nu}[t,y]$	mass-moment n^{ty}	1 [↻]	Isotropy-t,y : boost along t-y, φ_y
3	Mixed	Lorentz Boost $\Lambda^{\mu'}{}_{\nu} \to B^{\mu'}{}_{\nu}[t,z]$	mass-moment n^{tz}	1 [↻]	Isotropy-t,z : boost along t-z, φ_z
4	Spatial	Lorentz Rotate $\Lambda^{\mu'}{}_{\nu} \to R^{\mu'}{}_{\nu}[x,y]$	angular-momentum l^{xy}	1 [↻]	Isotropy-x,y : rotate about z, θ_z
5	Spatial	Lorentz Rotate $\Lambda^{\mu'}{}_{\nu} \to R^{\mu'}{}_{\nu}[x,z]$	angular-momentum l^{xz}	1 [↻]	Isotropy-x,z : rotate about y, θ_y
6	Spatial	Lorentz Rotate $\Lambda^{\mu'}{}_{\nu} \to R^{\mu'}{}_{\nu}[y,z]$	angular-momentum l^{yz}	1 [↻]	Isotropy-y,z : rotate about x, θ_x
7	Temporal	Translate Time $\Delta X^{\mu'} \to c\Delta t$	energy $E = cp^t$	1 [→]	Homogeneity-t : translate Δt
8	Spatial	Translate Space $\Delta X^{\mu'} \to \Delta x$	linear-momentum p^x	1 [→]	Homogeneity-x : translate Δx
9	Spatial	Translate Space $\Delta X^{\mu'} \to \Delta y$	linear-momentum p^y	1 [→]	Homogeneity-y : translate Δy
10	Spatial	Translate Space $\Delta X^{\mu'} \to \Delta z$	linear-momentum p^z	1 [→]	Homogeneity-z : translate Δz
				(10)	

Relativistic Particle · Vector Form: (scalar + 3-vectors): ***gathers the spatial parts into spatial 3-vectors*** {t,x,y,z} → {t,**x**}

#	Dim-Type	Isometry Operation	Conservation Law	# Parameters	Measure Symmetry
1	Mixed	Lorentz Boost $\Lambda^{\mu'}{}_{\nu} \to B^{\mu'}{}_{\nu}[t,\hat{\mathbf{n}}]$	3-mass-moment $\mathbf{n}=m^{0j}/c$	3-vector [↻]	Isotropy-t,$\hat{\mathbf{n}}$: boost along t-$\hat{\mathbf{n}}$, φ
2	Spatial	Lorentz Rotate $\Lambda^{\mu'}{}_{\nu}\to R^{\mu'}{}_{\nu}[\hat{\mathbf{n}}_1,\hat{\mathbf{n}}_2]$	3-angular-momentum $\mathbf{l}=\varepsilon_{ij}{}^{k}m^{ij}/2$	3-vector [↻]	Isotropy-$\hat{\mathbf{n}}_1$,$\hat{\mathbf{n}}_2$: rotate about $\hat{\mathbf{n}}_3$, θ
3	Temporal	Translate in Time $\Delta X^{\mu'} \to c\Delta t$	energy $E = cp^0$	1-scalar [→]	Homogeneity-t : in time Δt
4	Spatial	Translate in Space $\Delta X^{\mu'} \to \Delta\mathbf{x}$	3-linear-momentum $\mathbf{p}=p^k$	3-vector [→]	Homogeneity-**x** : in space $\Delta\mathbf{x}$
				(10)	

Relativistic Particle · Tensor Form: angular[↻] 4D Anti-Symmetric (2,0)-Tensor $M^{\mu\nu}$ + linear[→] 4D (1,0)-Tensor P^{μ} :{t,x}→{X}

#	Dim-Type	Isometry Operation	Conservation Law	# Parameters	Measure Symmetry
1	Mixed	Lorentz Transform $\Lambda^{\mu'}{}_{\nu}$	4-AngularMomentum $M^{\mu\nu}=X^{\mu}\wedge P^{\nu}$	6 [↻]	Isotropy-Spacetime $\omega_{\mu\nu}[\varphi,\theta]$ (✳)
2	Mixed	SpaceTime Translate $\Delta X^{\mu'}$	4-LinearMomentum P^{μ}	4 [→]	Homogeneity-Spacetime ΔX_{μ} [▦]
		($V^{\mu'} = \Lambda^{\mu'}{}_{\nu}V^{\nu} + \Delta V[\Delta X^{\mu'}]$) Linear-Affine Transform $V^{\mu'}=a*V^{\nu}+b$ ($V^{\mu'}V_{\mu'} = V^{\nu}V_{\nu} = \mathbf{V}\cdot\mathbf{V}$) Isometric = Same Measure	Noether's Theorem of Continuous Symmetries	(10) 6-bivector + 4-vector	Unitary Group U($\Lambda^{\mu'}{}_{\nu}$: $\Delta X^{\mu'}$) = e^[($^{i}/_{2\hbar}$)$\omega_{\mu\nu}M^{\mu\nu}$] : e^[($^{i}/_{\hbar}$)$\Delta X_{\mu}P^{\mu}$]

==

Relativistic Fluid Stress-Energy(Density) Tensor $T^{\mu\nu}$: 4D Symmetric (2,0)-Tensor, ***averages particles into a fluid description***

#	Dim-Type	Tensor Component, index	Tensor Component, type	# Parameters	Description
1	Temporal	t^{00}	energy-density $\rho_e = \rho_m c^2 = t^{00}$	1	the "dust" component
2	Mixed	t^{0j} and t^{i0} with $t^{0k} = t^{k0}$	3-heat-flux $\mathbf{q} = ct^{i0}$	3	the "heat-flux" components
3	Spatial	$t^{i=j} = p\delta^{ij}$	isotropic pressure $p = -(1/3)H_{\mu\nu}T^{\mu\nu}$	1	spatial diagonal components
4	Spatial	$t^{ij} = \Pi^{ij}$	anisotropic 6-viscous-shear Π^{ij}	5	only 5 independent, p removed
				(10)	

[⭮]
Lorentz Group ($\Lambda^{\mu}{}_{\nu}$) Symmetry → Conservation of 4-AngularMomentum $\boldsymbol{M} = M^{\mu\nu} = X^{\mu} \wedge P^{\nu} = [[m^{\mu\nu}]] = [[0,-c\mathbf{n}],[c\mathbf{n}^T,\mathbf{l}=\mathbf{x}\wedge\mathbf{p}]]$:Measure Isotropy
The spatial part: 3 Space-Space-Rotation ($\Lambda^{\mu}{}_{\nu}\to R^{\mu}{}_{\nu}$) Symmetry → Conservation of 3-angular-momentum $\mathbf{l} = l^k$ same all angular directions
The mixed part: 3 Space-Time-Boost ($\Lambda^{\mu}{}_{\nu}\to B^{\mu}{}_{\nu}$) Symmetry → Conservation of 3-mass-moment $\mathbf{n} = n^k$ θ,φ (∗)

[→]
Spacetime-Translation Group (ΔX^{μ}) Symmetry → Conservation of 4-LinearMomentum $\mathbf{P} = P^{\mu} = (p^{\mu}) = (p^0,p^k) = (E/c,\mathbf{p})$:Measure Homogeneity
The temporal part: 1 Time-Translation ($\Delta X^{\mu}\to\Delta x^0 = c\Delta t$) Symmetry → Conservation of energy $E = cp^0$ same all linear extent
The spatial part: 3 Space-Translation ($\Delta X^{\mu}\to\Delta x^k = \Delta\mathbf{x}$) Symmetry → Conservation of 3-momentum $\mathbf{p} = p^k$ ΔX [▦]

P: the 4-Momentum & **W**: the Pauli–Lubanski 4-Spin pseudovector. give $(\mathbf{P\cdot P})$ → mass (m_o) and $(\mathbf{W\cdot W})$ → spin (s_o), which are the two Casimir Invariants of the Poincaré Group, i.e. the quantities that commute with all generators of the Poincaré Group $\mathbf{R}^{1,3} \ltimes O(1,3)$.
Technically, $(\mathbf{P\cdot P})$ gives a 4-Momentum magnitude (moving mass) and $(\mathbf{W\cdot W})$ gives a 4-SpinMomentum magnitude (moving spin mass).

Notes on Poincaré SpaceTime Group, Tensor Linear Mapping, Lie Group, etc.:
SR 4-Vectors have a Poincaré Group linear mapping ($V^{\mu'} = \Lambda^{\mu'}{}_{\nu} V^{\nu} + \Delta V[\Delta X^{\mu'}]$) which preserves interval-magnitude: ($V^{\mu'}V_{\mu'} = V^{\nu}V_{\nu}$).
Poincaré = ISO(1,3) or $(\mathbb{R}\oplus\mathbb{R}^3)\rtimes O(1,3)$ or $\mathbf{R}^{1,3} \ltimes O(1,3)$ or $\mathbb{R}^4\rtimes SU_2(\mathbb{C})$. Isometry Group of Minkowski Space is the Poincaré Group.

The Poincaré Group is a Lie Group, and can be written as a Unitary Operation: $U(\Lambda^{\mu'}{}_{\nu} : \Delta X^{\mu'}) = e^{\wedge}[(^i/_{2\hbar})\omega_{\mu\nu}M^{\mu\nu}]$ [⭮] $e^{\wedge}[(^i/_{\hbar})\Delta X_{\mu}P^{\mu}]$ [→] with:
4-LinearMomentum P^{μ} [kg·m/s] as generator of SpaceTime-Translation-Transforms [→]
with 4-Displacement ΔX^{μ} [m] giving 1 temporal + 3 spatial = 4 linear displacements

4-AngularMomentum M^{μ} [kg·m²/s] as generator of Lorentz-Transforms [⭮]
with anti-symmetric Angular 4-Tensor $\omega^{\mu\nu}$ [1 or {rad}] giving 3 rotation angles + 3 boost hyper-angles = 6 angular displacements

$[\Delta X_{\mu}P^{\mu}]$ and $[\omega_{\mu\nu}M^{\mu\nu}]$ and (ħ) all have dimensional-units of [Action = kg·m²/s = J·s] :
Infinitesimal versions: additive $T^{\mu'} = 0^{\mu'} + \Delta X^{\mu'} + \ldots$: multiplicative $\Lambda^{\mu'}{}_{\nu} = \delta^{\mu'}{}_{\nu} + \omega^{\mu'}{}_{\nu} + \ldots$
===
4-AngularMomentum $M^{\mu\nu} = \mathbf{X}\wedge\mathbf{P} = X^{\mu}P^{\nu} - X^{\nu}P^{\mu}$
= Generator of Lorentz Transformations (6)
= { $\Lambda^{\mu'}{}_{\nu}\to R^{\mu'}{}_{\nu}$ Rotations (3) + $\Lambda^{\mu'}{}_{\nu}\to B^{\mu'}{}_{\nu}$ Boosts (3) }

Uses multiplicative identity: a = 1*a or $V^{\mu'} = \delta^{\mu'}{}_{\nu} V^{\nu}$
Non-motion $\Lambda^{\mu'}{}_{\nu}\to\delta^{\mu'}{}_{\nu}$ = Identity $I_{[4]}$: occurs at Trace[$\Lambda^{\mu'}{}_{\nu}$] = 4
===
4-LinearMomentum $P^{\mu} = \mathbf{P}$
= Generator of Translation Transformations (4)
= { $\Delta X^{\mu'}\to(c\Delta t,\mathbf{0})$ Time (1) + $\Delta X^{\mu'}\to(0,\Delta\mathbf{x})$ Space (3) }

Uses additive identity: a = a+0 or $V^{\mu'} = V^{\nu} + 0^{\mu'}$
Non-motion $\Delta x^{\mu'}\to(0,\mathbf{0}) = 0^{\mu'}$ = 4-Zero : 4-Zero remains 4-Zero in all LT frames
===
Symmetric:AntiSymmetric Tensor decomposition $\{T^{\mu\nu} = S^{\mu\nu} + A^{\mu\nu}\}$, with $S^{\mu\nu} = (T^{\mu\nu}+T^{\nu\mu})/2$ and $A^{\mu\nu} = (T^{\mu\nu}-T^{\nu\mu})/2$
Tensor Contraction of Symmetric with AntiSymmetric yields zero $\{S^{\mu\nu}A_{\mu\nu} = 0\}$, from { $S^{\mu\nu} = +S^{\nu\mu}$ } and { $A^{\mu\nu} = -A^{\nu\mu}$ } parts of { $T^{\mu\nu}$ }
Proof: 10 components + 6 components = 16 comps
$S^{\mu\nu}A_{\mu\nu}$ ={swapping dummy indices}→ $S^{\nu\mu}A_{\nu\mu} = (S^{\nu\mu})(A_{\nu\mu}) = (+S^{\mu\nu})(A_{\nu\mu}) = (+S^{\mu\nu})(-A_{\mu\nu}) = -S^{\mu\nu}A_{\mu\nu} = 0$, since { C = –C = 0 }
The Symmetric Tensor is further decomposed into an Isotropic part $S_{iso}{}^{\mu\nu} = (S^{\alpha}{}_{\alpha}/4)\eta^{\mu\nu}$ and zero-trace Anisotropic part $S_{aniso}{}^{\mu\nu} = S^{\mu\nu}-S_{iso}{}^{\mu\nu}$
So, $\{T^{\mu\nu} = S_{iso}{}^{\mu\nu} + S_{aniso}{}^{\mu\nu} + A^{\mu\nu}\}$ This is manifestly invariant: The Poincaré Group Symmetry operations respect these decompositions, meaning that boosts, rotations, etc. don't intermix them, unlike the (temporal+mixed+spatial)-splittings, which can get intermixed.
==
Lorentz Transform $\Lambda^{\mu'}{}_{\nu}\to B^{\mu'}{}_{\nu}$ ($\boldsymbol{\beta}$) or $B^{\mu'}{}_{\nu}$ ($\varphi_{hyperangle},\hat{\mathbf{n}}$) Boost : Symmetric Mixed 4-Tensor $B^T = B$: $B^{-1} = B[-\boldsymbol{\beta}]$: 3 parameters
$[\gamma \quad , \quad -\gamma\beta^0{}_j \quad]$
$[-\gamma\beta^i{}_0 , \ (\gamma-1)\beta^i\beta_j/(\boldsymbol{\beta\cdot\beta})+\delta^i{}_j]$

Trace of Boost = {4..Infinity}
note hyperbolic form using: γ = cosh(φ) : γβ = sinh(φ) : β = tanh(φ)
==
Lorentz Transform $\Lambda^{\mu'}{}_{\nu}\to R^{\mu'}{}_{\nu}$ ($\theta_{angle},\hat{\mathbf{n}}$) Rotation : Non-symmetric Mixed 4-Tensor : Orthogonal $R^T = R^{-1}$: $R^{-1} = R[-\theta]$: 3 parameters
$[\ 1 \ , \quad 0^0{}_j \quad]$
$[0^i{}_0 , (\ \delta^i{}_j-n^in_j\ )\cos(\theta)-(\ \varepsilon^i{}_{jk}n^k\ )\sin(\theta)+n^in_j]$

Trace of Rotation = {0..4}
==
Lorentz Transform $\Lambda^{\mu'}{}_{\nu}\to I^{\mu'}{}_{\nu} = \delta^{\mu'}{}_{\nu}$ = Diag[1,1,1,1] Identity : Symmetric Mixed 4-Tensor : $B[\boldsymbol{\beta}=\mathbf{0}] = R[\theta=0]$: 0 parameters
$[\ 1 \ , 0^0{}_j]$
$[0^i{}_0 , \ \delta^i{}_j]$

Trace of Identity = {4}
The Boost and Rotation forms "meet" each other at the Identity Transform, at the Trace[Lorentz] = 4.
==
There are also various combinations of Discrete Flips $\Lambda^{\mu'}{}_{\nu}\to F^{\mu'}{}_{\nu}$ of coordinates: Trace = {-4,-2,0,2,4}
which when paired (ex. t→-t & x→-x , y=y, z=z) make Proper Discrete Transforms Det = +1.
Also included { Spatial-Parity, Time-Reversal } Det = -1, and { Charge-Conjugation } Det = +1

$[\ 1 \ , 0^0{}_j]$ $[-1 \ , 0^0{}_j]$ $[-1 \ , 0^0{}_j]$
$[0^i{}_0 , -\delta^i{}_j]$ $[0^i{}_0 , \ \delta^i{}_j]$ $[0^i{}_0 , -\delta^i{}_j]$

4D-Tensor Theory of Measurements (SR Poincaré Invariance = Lorentz $\Lambda^{\mu'}{}_{\nu}$ Invariance + SpaceTime Translation $\Delta X^{\mu'}$ Invariance):
SR 4-Vectors have a Poincaré Group 4D linear mapping {technically a linear-affine transformation due to the additive constant}
($V^{\mu'} = \Lambda^{\mu'}{}_{\nu} V^{\nu} + \Delta V[\Delta X^{\mu'}]$) which preserves interval-magnitude: ($V^{\mu'} V_{\mu'} = V^{\nu} V_{\nu} = \mathbf{V}\cdot\mathbf{V}$), which is a 4-Scalar calculated from 4-Vectors.

This idea basically says the following:
A measurement made in one reference frame has an affine relationship to the same measurement made in a different reference frame.
In other words, it is a linear relation (a) with a possible additive constant (b) mapping on (**X**): **X**'=a**X**+b *{the equation of a straight line}*
This means that there are certain transformations:symmetries that one can do and still get the same invariant measurement interval.

Active SR Transformations (the system being measured is changed in a certain way, the reference coordinate frame is not changed):
Rotate the system of objects. 3 Ex. Pick an angle:axis and rotate whole experiment by that amount. $\Lambda^{\mu'}{}_{\nu} \rightarrow R^{\mu'}{}_{\nu}(\theta_{angle} : \hat{\mathbf{n}})$
Boost the system of objects. 3 Ex. Have the whole experiment move uniformly on a linear track. $\Lambda^{\mu'}{}_{\nu} \rightarrow B^{\mu'}{}_{\nu}(\boldsymbol{\beta})$ or $B^{\mu'}{}_{\nu}(\varphi_{hyperangle} : \hat{\mathbf{n}})$
Translate the system of objects in space. 3 Ex. Move everything 2 meters to the left. $\Delta X^{\mu'} \rightarrow \Delta\mathbf{x}$
Translate the system of objects in time. 1 Ex. Wait 3 minutes and then measure. $\Delta X^{\mu'} \rightarrow \Delta ct$

Passive SR Transformations (the system being measured is not changed, the reference coordinate frame is changed in a certain way):
Rotate the reference coordinate frame. Ex. Rotate your measuring rods about some axis, then do measurement.
Boost the reference coordinate frame. Ex. Have the measuring rods:clock uniformly move on a linear track, then do measurement.
Translate the reference coordinate frame in space. Ex. Move the measuring rods 5 feet south, then do measurement.
Translate the reference coordinate frame in time. Ex. Take pic of object. Wait 7 seconds on clock, then do measurement on pic.

Note: The difference between active:passive transforms is whether the measured object or the measuring system gets "changed".
The sameness is the result of the measurement. All reference frame lead to the same measurement invariant.

==

There are (10) one-parameter groups can be expressed directly as exponentials of the physical generators.
Poincaré Algebra is the Lie Algebra of the Poincaré Group. The Isometry Group of Minkowski Space is the Poincaré Group.
These are Noether Symmetries. A physically continuous invariance leads to a conserved physical quantity, a conserved current and vice-versa.

$U[\ I, (a^0,0)\ ] = e\hat{}(ia^0 H) = e\hat{}(ia^0 p^0)$: (1) Hamiltonian (Energy) H = Linear Temporal Momentum
$U[\ I, (0,\lambda\hat{\mathbf{a}})\ ] = e\hat{}(-i\lambda\hat{\mathbf{a}}\cdot\mathbf{p})$: (3) Linear Spatial Momentum **p**
$U[\ \Lambda(i\lambda\hat{\boldsymbol{\theta}}/2), 0\ ] = e\hat{}(i\lambda\hat{\boldsymbol{\theta}}\cdot\mathbf{j})$: (3) Angular Spatial Momentum **j** = **l**
$U[\ \Lambda(\lambda\hat{\boldsymbol{\varphi}}/2), 0\ ] = e\hat{}(i\lambda\hat{\boldsymbol{\varphi}}\cdot\mathbf{k})$: (3) Dynamic SpaceTime Mass Moment **k** = **n**
The Poincaré Algebra is the Lie Algebra of the Poincaré Group:
Total of {1+3+3+3 = (1+3)+(3+3) = 4+6 = 10} Invariances from Poincaré Symmetry
$U(\Lambda^{\mu'}{}_{\nu} : \Delta X^{\mu'}) = e\hat{}[(^{i}/_{2\hbar})\omega_{\mu\nu}M^{\mu\nu}] : e\hat{}[(^{i}/_{\hbar})\Delta X_{\mu}P^{\mu}]$

Conservation of 4-LinearMomentum $\mathbf{P} = P^{\mu}$: (1 + 3) = (4) Laws {Linear [→]} = Measure Homogeneity of Spacetime [▦]
Conservation of scalar energy E (temporal) = Invariance of Time Translation — same all linear extent $\Delta\mathbf{X}$
Conservation of 3-vector 3-momentum $\mathbf{p} = p^k$ (spatial) = Invariance of Space Translation

Conservation of 4-AngularMomentum $M^{\mu\nu} = \mathbf{X}\hat{}\mathbf{P} = X^{\mu}P^{\nu} - X^{\nu}P^{\mu}$: (3 + 3) = (6) Laws {Angular [↻]} = Measure Isotropy of Spacetime (∗)
Conservation of relativistic 3-mass-moment $\mathbf{n} = n^k$ (temporal-spatial) = Invariance of Boost — same all angular directions θ,φ
Conservation of angular 3-momentum $\mathbf{l} = \mathbf{x}\hat{}\mathbf{p} = l^k$ (spatial-spatial) = Invariance of Rotation

(10) Total Individual Conservation Laws

4D Tensor Considerations, the (10) independent parameters can take several different tensorial forms:

==

{1} Symmetric 4D (2,0)-Tensor:(10) independent parameters, with tensor symmetry giving the property $[T^{\nu\mu}] = +[T^{\mu\nu}]$

4D Stress-Energy(Density) Tensor $T^{\mu\nu}$: has (10) independent parameters $\{\rho_e,p,\mathbf{q},\mathbf{\Pi}\}=\{1,1,3,5\}$
energy-density ρ_e (1), isotropic pressure p (1), heat-flux $\mathbf{q}$ (3), anisotropic viscous stress $\mathbf{\Pi}$ (5)
$T^{\mu\nu} =$
$[\rho_e=\rho_m c^2, \mathbf{q}/c]$
$[\mathbf{q}^T/c, p\delta^{ij} + \Pi^{ij}]$
It is useful for describing relativistic fluids (☁). Technically a Stress-EnergyDensity, but often referred to as Stress-Energy

==

{1} Anti-Symmetric 4D (2,0)-Tensor:(6) , with tensor anti-symmetry giving the property $[T^{\nu\mu}] = -[T^{\mu\nu}]$
+
{1} 4-Vector = 4D (1,0)-Tensor:(4)

4-AngularMomentum $M^{\mu\nu}$ [↻]: a 4D-bivector with 6 independent parameters, 3-angular-momentum $\mathbf{l}$ & 3-mass-moment $\mathbf{n}$
4-LinearMomentum P^{μ} [→]: a 4D-vector with 4 independent parameters, 1-energy E & 3-momentum $\mathbf{p}$
for a total of (10) independent parameters $\{\mathbf{l},\mathbf{n},E,\mathbf{p}\}=\{3+3+1+3\}$
There are well known conservation laws associated with these.
They are useful for describing relativistic particles (·).
Bivector ($\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$) ~ Skew-Symmetric Matrix ($\mathbf{A}^T = -\mathbf{A}$) ~ 2-index Anti-Symmetric Tensor ($A^{\mu\nu} = -A^{\nu\mu}$)
parameters = $n(n-1)/2$, with n=4, gives 4(4-1)/2=6 parameters
Also EM: using the Faraday EM Tensor, leading to (10) independent parameters of Maxwell Equations and Lorentz Force Equation

==

{3} 4-Vectors = 4D (1,0)-Tensors:(4) components each, which would normally give (12) independent parameters.

However, there is a universal constant (lightspeed c) which reduces this to (10) independent parameters, using time derivative twice.
4-Position $R^{\mu} = (ct,\mathbf{r})$ has 4 independent parameters, t & $\mathbf{r}$
4-Velocity $U^{\mu} = \gamma(c,\mathbf{u}) = (d/d\tau)R^{\mu}$ has 3 independent parameters $\mathbf{u}$, due to the constraint $(\mathbf{U}\cdot\mathbf{U}) = c^2$
4-Acceleration $A^{\mu} = \gamma(c\gamma', \gamma'\mathbf{u}+\gamma\mathbf{a}) = (d/d\tau)U^{\mu}$ has 3 or 6 independent parameters, depending on how you examine it.
It is based on 3-velocity $\mathbf{u}$ and 3-acceleration $\mathbf{a}$, and 4→3 due to the constraint $(\mathbf{U}\cdot\mathbf{A}) = 0 \leftrightarrow \mathbf{U}\perp\mathbf{A}$
Thus, the three 4-Vectors taken as a group have only (10) independent parameters: $\{t,\mathbf{r},\mathbf{u},\mathbf{a}\}=\{1+3+3+3\}$.
There are 3*4=12 parameters for the three 4-Vectors, but 2 constraint equations { $(\mathbf{U}\cdot\mathbf{U}) = c^2$: $(\mathbf{U}\cdot\mathbf{A}) = 0$ }, leaving just (10).

==

$T^{\mu\nu}$ = General 4D (2,0)-Tensor = with 16 independent components
$[t^{00}\ t^{01}\ t^{02}\ t^{03}]$
$[t^{10}\ t^{11}\ t^{12}\ t^{13}]$
$[t^{20}\ t^{21}\ t^{22}\ t^{23}]$
$[t^{30}\ t^{31}\ t^{32}\ t^{33}]$

$S^{\mu\nu}$ = Symmetric 4D (2,0)-Tensor = with 10 independent components, due to $S^{\mu\nu} = +S^{\nu\mu}$
$[s^{00}\ \ s^{01}\ \ s^{02}\ \ s^{03}]$
$[+s^{01}\ \ s^{11}\ \ s^{12}\ \ s^{13}]$
$[+s^{02}\ +s^{12}\ +s^{22}\ \ s^{23}]$
$[+s^{03}\ +s^{13}\ +s^{23}\ +s^{33}]$

$A^{\mu\nu}$ = AntiSymmetric 4D (2,0)-Tensor = with 6 independent components, due to $A^{\mu\nu} = -A^{\nu\mu}$
$[\ 0\ \ a^{01}\ \ a^{02}\ \ a^{03}]$
$[-a^{01}\ \ 0\ \ a^{12}\ \ a^{13}]$
$[-a^{02}\ -a^{12}\ \ 0\ \ a^{23}]$
$[-a^{03}\ -a^{13}\ -a^{23}\ \ 0\ ]$

V^{μ} = General 4D (1,0)-Tensor = 4-Vector = with 4 independent components
(v^0,v^1,v^2,v^3)

S = General 4D (0,0)-Tensor = 4-Scalar = with 1 independent component
(s_o)

{1} Symmetric 4D (2,0)-Tensor
Full Relativistic Fluid Stress-Energy(Density) 4-Tensor,$_{\text{Symmetric}}$: $T^{\alpha\beta} = +T^{\beta\alpha}$: (10) Independent parameters
$T_{RelFluid}^{\mu\nu} = (\rho_{eo})V^{\mu\nu} + (-p_o)H^{\mu\nu} + (\bar{T}^{\mu}H^{\nu}{}_{\sigma}Q^{\sigma} + Q^{\sigma}H^{\mu}{}_{\sigma}\bar{T}^{\nu})/c + \Pi^{\mu\nu}$ [kg/m·s² = J/m³ = N/m² = Pa]
→
$[T^{tt}\ \ T^{tx}\ \ T^{ty}\ \ T^{tz}]$
$[T^{xt}\ \ T^{xx}\ \ T^{xy}\ \ T^{xz}]$
$[T^{yt}\ \ T^{yx}\ \ T^{yy}\ \ T^{yz}]$
$[T^{zt}\ \ T^{zx}\ \ T^{zy}\ \ T^{zz}]$
=
$[\ \rho_e{=}\rho_m c^2\ \ q^{01}/c\ \ q^{02}/c\ \ q^{03}/c\ ]$
$[\ q^{10}/c\ \ p + \Pi^{11}\ \ \Pi^{12}\ \ \Pi^{13}\ ]$
$[\ q^{20}/c\ \ \Pi^{21}\ \ p + \Pi^{22}\ \ \Pi^{23}\ ]$
$[\ q^{30}/c\ \ \Pi^{31}\ \ \Pi^{32}\ \ p + \Pi^{33}]$
=
$[\ \rho_e{=}\rho_m c^2\ \ q^x/c\ \ q^y/c\ \ q^z/c\ ]$
$[\ q^x/c\ \ p + \Pi^{xx}\ \ \Pi^{xy}\ \ \Pi^{xz}\ ]$
$[\ q^y/c\ \ \Pi^{yx}\ \ p + \Pi^{yy}\ \ \Pi^{yz}\ ]$
$[\ q^z/c\ \ \Pi^{zx}\ \ \Pi^{zy}\ \ p + \Pi^{zz}]$
=
$[\rho_e{=}\rho_m c^2,\ \mathbf{q}/c\ ]$
$[\mathbf{q}^T/c,\ \ p\delta^{ij} + \Pi^{ij}]$
==

(temporal:mixed:spatial) splitting
1 Temporal:Temporal EnergyDensity $(\rho_{eo}){=}V_{\mu\nu}T^{\mu\nu}$
3 Temporal:Spatial HeatEnergy Flux $(Q^{\mu}){=}c\bar{T}_{\nu}T^{\mu\nu}$
1 Spatial:Spatial Isotropic Pressure $(p_o){=}(-1/3)H_{\mu\nu}T^{\mu\nu}$
5 Spatial:Spatial Anisotropic Viscous Stress $(\Pi^{\mu\nu}){=}H^{\mu}{}_{\alpha}H^{\nu}{}_{\beta}T^{\alpha\beta} + (p_o)H^{\mu\nu}$
(10) Total Independent Components

Full Relativistic Fluid Stress-Energy(Density)
$T^{\mu\nu} = (\rho_{eo})V^{\mu\nu} + (-p_o)H^{\mu\nu} + (\bar{T}^{\mu}H^{\nu}{}_{\sigma}Q^{\sigma} + Q^{\sigma}H^{\mu}{}_{\sigma}\bar{T}^{\nu})/c + \Pi^{\mu\nu} \rightarrow [[\rho_e, q^{0j}/c\ ],[q^{i0}/c, p\delta^{ij} + \Pi^{ij}]]_{\{MCRF\}}$
(ρ_{eo}) = (Temporal) EnergyDensity 4-Scalar : 1 independent component
(p_o) = (Spatial) Isotropic Pressure 4-Scalar : 1 independent component
$(\bar{T}^{\mu})$ = UnitTemporal 4-Vector
(Q^{μ}) = HeatEnergyFlux 4-Vector w/ $Q^{\mu}\bar{T}_{\mu} = 0$: 3 independent components aka. MomentumDensity
$(\Pi^{\mu\nu})$ = ViscousShear 4-Tensor w/ $\Pi^{\mu\nu}\bar{T}_{\mu} = 0^{\nu}$: 5 indep. components aka. AnisotropicStress (traceless $Tr[\Pi^{\mu\nu}]=0$ and $\Pi^{\mu\nu}{=}H^{\mu}{}_{\rho}H^{\nu}{}_{\sigma}\Pi^{\sigma\rho}$)
$(V^{\mu\nu})$ = (Temporal) (V)ertical Projection 4-Tensor
$(H^{\mu\nu})$ = (Spatial) (H)orizontal Projection 4-Tensor

The Full ViscousShear Tensor has 6 total components = 1 isotropic pressure + 5 anisotropic shears
There is a decomposition into an Isotropic (with trace) and Anisotropic (without trace = traceless = zero trace) part.
The Isotropic part is the Trace of the Tensor, and the Anisotropic part is the (Full – Isotropic) part.

We can do this because the Trace operation is itself a single component (scalar) tensor invariant.
Define Isotropic $I^{\mu\nu} = (\eta^{\mu\nu}/4)Trace[T^{\mu\nu}]$, with the $Trace[T^{\mu\nu}] = \eta_{\mu\nu}\ T^{\mu\nu}$. $Trace[I^{\mu\nu}] = \eta_{\mu\nu}\ I^{\mu\nu} = \eta_{\mu\nu}\ (\eta^{\mu\nu}/4)Trace[T^{\mu\nu}] = Trace[T^{\mu\nu}]$
Define Anisotropic $A^{\mu\nu} = T^{\mu\nu} - I^{\mu\nu}$. Since $Trace[T^{\mu\nu}] = Trace[I^{\mu\nu}]$, $Trace[A^{\mu\nu}] = 0$
$T^{\mu\nu} = I^{\mu\nu} + A^{\mu\nu} = I^{\mu\nu} + (T^{\mu\nu} - I^{\mu\nu}) = T^{\mu\nu}$
Thus, the Anisotropic part has 5 independent components, and the Isotropic part has 1 independent component

$F_{den}{}^{\mu} = \partial_{\nu}T^{\mu\nu}$ Independent Components: ∂_{ν} (4), $T^{\mu\nu}$ (10), Eqns (-4) = (10) Dependent Components: $F_{den}{}^{\mu}$ (4)

{1} Anti-Symmetric 4D (2,0)-Tensor + {1} 4-Vector = 4D (1,0)-Tensor

==

[→] 4-LinearMomentum 4-Vector : 4 Independent parameters

$P^\mu = (E/c=mc, \mathbf{p}=m\mathbf{u}) = \mathbf{P}$ [kg·m/s = N·s]

==

[↻] 4-AngularMomentum 4-Tensor,$_{Antisymmetric}$: $M^{\alpha\beta} = -M^{\beta\alpha}$: 6 Independent parameters

$M^{\alpha\beta} = X^\alpha P^\beta - X^\beta P^\alpha = \mathbf{X}\wedge\mathbf{P}$ [kg·m²/s = N·m·s = J·s = Action]

→

$$\begin{bmatrix} M^{tt} & M^{tx} & M^{ty} & M^{tz} \\ M^{xt} & M^{xx} & M^{xy} & M^{xz} \\ M^{yt} & M^{yx} & M^{yy} & M^{yz} \\ M^{zt} & M^{zx} & M^{zy} & M^{zz} \end{bmatrix}$$

=

$$\begin{bmatrix} 0 & x^0p^1-x^1p^0 & x^0p^2-x^2p^0 & x^0p^3-x^3p^0 \\ x^1p^0-x^0p^1 & 0 & x^1p^2-x^2p^1 & x^1p^3-x^3p^1 \\ x^2p^0-x^0p^2 & x^2p^1-x^1p^2 & 0 & x^2p^3-x^3p^2 \\ x^3p^0-x^0p^3 & x^3p^1-x^1p^3 & x^3p^2-x^2p^3 & 0 \end{bmatrix}$$

=

$$\begin{bmatrix} 0 & ctp^x-xE/c & ctp^y-yE/c & ctp^z-zE/c \\ xE/c-ctp^x & 0 & xp^y-yp^x & xp^z-zp^x \\ yE/c-ctp^y & yp^x-xp^y & 0 & yp^z-zp^y \\ zE/c-ctp^z & zp^x-xp^z & zp^y-yp^z & 0 \end{bmatrix}$$

=

$$\begin{bmatrix} 0 & c(tp^x-xm) & c(tp^y-ym) & c(tp^z-zm) \\ c(xm-tp^x) & 0 & xp^y-yp^x & xp^z-zp^x \\ c(ym-tp^y) & yp^x-xp^y & 0 & yp^z-zp^y \\ c(zm-tp^z) & zp^x-xp^z & zp^y-yp^z & 0 \end{bmatrix}$$

=

$$\begin{bmatrix} 0 & -cn^x & -cn^y & -cn^z \\ +cn^x & 0 & +l^z & -l^y \\ +cn^y & -l^z & 0 & +l^x \\ +cn^z & +l^y & -l^x & 0 \end{bmatrix}$$

=

$$\begin{bmatrix} 0 & , -cn^{0j} \\ +cn^{i0}, & \varepsilon^{ij}{}_k l^k \end{bmatrix}$$

=

$$\begin{bmatrix} 0 & , -c\mathbf{n} \\ +c\mathbf{n}^T, & \mathbf{x}\wedge\mathbf{p} \end{bmatrix}$$

==

A particle can be part of a system, with 4-AngularMomentum $M^{\mu\nu}$ (6 parameters).
Then, the whole system can be moving uniformly with a 4-Momentum $P_{linear}{}^\mu$ (4 parameters).
Total is (10) Independent parameters.
A simple example might be a 2-bola, 2 equal point-masses swinging around a central point, held by mutual tension.
Then, the whole system, defined by the central point, could be moving in a linear direction.
There are (10) independent Conservation Laws composed of LinearMomentum [→] $P_{linear}{}^\mu$ and AngularMomentum [↻] $M^{\mu\nu}$ parts.
4-LinearMomentum $P_{linear}{}^\mu = (E_{linear}/c, \mathbf{p}_{linear})$ has (4) independent linear parameters.
4-Position $R^\mu = (ct, \mathbf{r})$ has (4) independent parameters.
4-RotationalMomentum $P_{rotational}{}^\mu = (E_{rotational}/c, \mathbf{p}_{rotational})$ has (4) independent parameters. (not the 4-AngularMomentum 4-Tensor)
This would normally give a total of (8) independent angular parameters, However, in 4D, there are (2) tensor invariants for the combination:
4-AngularMomentum $M^{\mu\nu} = R^\mu \wedge P_{rotational}{}^\nu = R^\mu P_{rotational}{}^\nu - R^\nu P_{rotational}{}^\mu$: This leaves (6) total independent angular parameters.
Thus, [→] [↻]
Conservation of 4-LinearMomentum $P_{linear}{}^\mu$ (4) + Conservation of 4-AngularMomentum $M^{\mu\nu} = R^\mu \wedge P_{rotational}{}^\nu$ (6) = (10) independent.

{1} Anti-Symmetric 4D (2,0)-Tensor + {1} 4-Vector = 4D (1,0)-Tensor, in EM

4-Gradient $\partial^\mu = (\partial_t/c, -\nabla) = \partial$ [1/m]
4-Velocity $U^\mu = \gamma(c, \mathbf{u}) = \mathbf{U}$ [m/s]
4-(EM)VectorPotential $A^\mu = (\varphi/c, \mathbf{a}) = \mathbf{A}$ [T·m]
4-CurrentDensity $J^\mu = (\rho c, \mathbf{j}) = \mathbf{J}$ [A/m²] = [C/m²·s] =4-ChargeFlux, a moving charge density [C/m³ · m/s]
4-Velocity $U^\mu = \gamma(c, \mathbf{u}) = \mathbf{U}$ [m/s]
4-Force $F^\mu = \gamma(\dot{E}/c, \mathbf{f}) = \mathbf{F}$ [kg·m/s²]

==

Faraday EM 4-Tensor,$_{\text{Antisymmetric}}$: $F^{\alpha\beta} = -F^{\beta\alpha}$: 6 Independent parameters = ∂ (4) ; $\mathbf{A}$ (4) ; Tensor Invariants(-2) ; $F^{\alpha\beta}$ (6) ; Eqns (-6)

$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha = \partial \wedge \mathbf{A}$ [kg·m²/s = N·m·s = J·s = Action] 4 + 4 – 2 + 6 – 6 = 6

→

$$\begin{bmatrix} F^{tt} & F^{tx} & F^{ty} & F^{tz} \\ F^{xt} & F^{xx} & F^{xy} & F^{xz} \\ F^{yt} & F^{yx} & F^{yy} & F^{yz} \\ F^{zt} & F^{zx} & F^{zy} & F^{zz} \end{bmatrix}$$

=

$$\begin{bmatrix} 0 & \partial^0 a^1 - \partial^1 a^0 & \partial^0 a^2 - \partial^2 a^0 & \partial^0 a^3 - \partial^3 a^0 \\ \partial^1 a^0 - \partial^0 a^1 & 0 & \partial^1 a^2 - \partial^2 a^1 & \partial^1 a^3 - \partial^3 a^1 \\ \partial^2 a^0 - \partial^0 a^2 & \partial^2 a^1 - \partial^1 a^2 & 0 & \partial^2 a^3 - \partial^3 a^2 \\ \partial^3 a^0 - \partial^0 a^3 & \partial^3 a^1 - \partial^1 a^3 & \partial^3 a^2 - \partial^2 a^3 & 0 \end{bmatrix}$$

=

$$\begin{bmatrix} 0 & (\partial^t a^x + \partial^x \varphi)/c & (\partial^t a^y + \partial^y \varphi)/c & (\partial^t a^z + \partial^z \varphi)/c \\ (-\partial^x \varphi - \partial^t a^x)/c & 0 & -\partial^x a^y + \partial^y a^x & -\partial^x a^z + \partial^z a^x \\ (-\partial^y \varphi - \partial^t a^y)/c & -\partial^y a^x + \partial^x a^y & 0 & -\partial^y a^z + \partial^z a^y \\ (-\partial^z \varphi - \partial^t a^z)/c & -\partial^z a^x + \partial^x a^z & -\partial^z a^y + \partial^y a^z & 0 \end{bmatrix}$$

=

$$\begin{bmatrix} 0 & -e^x/c & -e^y/c & -e^z/c \\ +e^x/c & 0 & -b^z & +b^y \\ +e^y/c & +b^z & 0 & -b^x \\ +e^z/c & -b^y & +b^x & 0 \end{bmatrix}$$

=

$$\begin{bmatrix} 0 & , -e^{0j}/c \\ +e^{i0}/c, & \varepsilon^{ij}{}_k b^k \end{bmatrix}$$

=

$$\begin{bmatrix} 0 & , -c\mathbf{e} \\ +c\mathbf{e}^T, & \partial \wedge \mathbf{a} \end{bmatrix}$$

==

$\partial_\alpha F^{\alpha\beta} = \mu_o J^\beta$: Gauss-Ampere Law = Inhomogeneous Maxwell EM Equations : Two factors of $\partial_\mathbf{r}$ in EoM: $[\partial_\alpha F^{\alpha\beta}] = [\partial_\alpha(\partial^\alpha A^\beta - \partial^\beta A^\alpha)]$

Parameters from: ∂_α (4) ; $F^{\alpha\beta}$ (6) ; μ_o (0) a universal constant ; J^β (4) ; Eqns (–4) Due to equations, the J^β (4) are dependent parameters.

4 + 6 + 0 + 4 – 4 = (10) Total Independent Parameters

Also, a totally beautiful result emerges: The Conservation of Charge.

$\partial_\beta \partial_\alpha F^{\alpha\beta} = \mu_o \partial_\beta J^\beta = 0 \rightarrow (\partial \cdot \mathbf{J}) = 0$: because $(\partial_\beta \partial_\alpha)$ is Symmetric, $(F^{\alpha\beta})$ is Anti-symmetric

$S^{\mu\nu} A_{\mu\nu}$ ={swapping dummy indices}→ $S^{\nu\mu} A_{\nu\mu} = (S^{\nu\mu})(A_{\nu\mu}) = (+S^{\mu\nu})(A_{\nu\mu}) = (+S^{\mu\nu})(-A_{\mu\nu}) = -S^{\mu\nu} A_{\mu\nu} = 0$, since { C = –C = 0 }

$F^\beta = q U_\alpha F^{\alpha\beta}$: Lorentz EM Force Equation : $n_o F^\beta = n_o q U_\alpha F^{\alpha\beta} = F_{den}{}^\beta = \rho_o U_\alpha F^{\alpha\beta} = J_\alpha F^{\alpha\beta}$: $F_{den}{}^\beta = J_\alpha F^{\alpha\beta}$

q depends on the particle type { -e, -(2/3)e ,-(1/3)e, 0, (1/3)e, (2/3)e, e }

Parameters from: F^β (4) ; q (1) ; U_α (3) ; $F^{\alpha\beta}$ (6) ; Eqns (–4) Due to equations, the F^β (4) are dependent parameters.

4 + 1 + 3 + 6 – 4 = (10) Total Independent Parameters

$\partial^\alpha \varepsilon_\alpha{}^\beta{}_{\gamma\delta} F^{\gamma\delta} = 0^\beta$: Gauss-Faraday Law : since various terms cancel, this gives $+\partial^\sigma F^{\mu\nu} + \partial^\mu F^{\nu\sigma} + \partial^\nu F^{\sigma\mu} = 0$

6 Independent Parameters from: ∂^α (4) ; $F^{\gamma\delta}$ (6) ; $\varepsilon_\alpha{}^\beta{}_{\gamma\delta}$ (0) a constant ; Eqns (–4)

4 + 6 – 4 = 6 Total Independent Parameters

{3} 4-Vectors= 4D (1,0)-Tensors

Relativistic Particle, Tensor Form: 3 SR 4D (1,0)-Tensors = {4-Position, 4-Velocity, 4-Acceleration}

4-Vectors = 4D (1,0)-Tensors:

4-Position $R^\mu = (ct,\mathbf{r}) = \mathbf{R} \in$ <event> [m] $(ct,\mathbf{r}) \to (ct,x,y,z)$ only Lorentz, not Poincaré Invariant

4-Velocity $U^\mu = \gamma(c,\mathbf{u}) = \mathbf{U} = d\mathbf{R}/d\tau$ [m/s] Lorentz Gamma Factor $\gamma = 1/\sqrt{[1-(u/c)^2]} = dt/d\tau$

4-Acceleration $A^\mu = \gamma(c\gamma',\gamma'\mathbf{u}+\gamma\mathbf{a}) = \mathbf{A} = d\mathbf{U}/d\tau = d^2\mathbf{R}/d\tau^2$ [m/s^2] $A^\mu = (\gamma^4(\mathbf{a}\cdot\mathbf{u})/c,\gamma^4(\mathbf{a}\cdot\mathbf{u})\mathbf{u}/c^2+\gamma^2\mathbf{a}) = \gamma^2(\gamma^2(\mathbf{a}\cdot\mathbf{u})/c,\gamma^2(\mathbf{a}\cdot\mathbf{u})\mathbf{u}/c^2+\mathbf{a})$

Examine the total # of independent components:parameters:
Some are Degrees of Freedom (DoF): spacetime variables which change along a worldline
Some are Constraints: those parameters which force a particle upon a particular path or worldline

4-Position gives (4) independent parameters with t:(1) and $\mathbf{r}$:(3)
4-Velocity gives (3) independent parameters with $\mathbf{u}$:(3), since γ is a function of $\mathbf{u}$
The constraint is $\mathbf{U}\cdot\mathbf{U} = c^2$
4-Acceleration gives (3) in this context, the only new one is $\mathbf{a}$:(3), γ is a function of $\mathbf{u}$, and $\gamma' = \gamma^3(\mathbf{a}\cdot\mathbf{u})/c^2$ is a function of $\mathbf{a}$ and $\mathbf{u}$
The constraint is $\mathbf{U}\cdot\mathbf{A} = 0 \leftrightarrow \mathbf{U}\perp\mathbf{A}$
Thus, there are total of (10) independent parameters: {t,$\mathbf{r}$,$\mathbf{u}$,$\mathbf{a}$}={1+3+3+3} which allow the generalized particle dynamics.
This is derived by using the ProperTimeDerivative twice ($d/d\tau$).

==

Likewise there are (10) constants of the motion:
Choose an inertial free-particle frame under which the 3-acceleration $\mathbf{a} = \mathbf{a}_{init} = \mathbf{0}$. This is 3 constraints and 3 constants.
This gives $\gamma \to 1$, $\gamma' = \gamma^3(\mathbf{a}\cdot\mathbf{u})/c^2 \to 0$, which in turn gives $\mathbf{A} = A^\mu = \gamma(c\gamma',\gamma'\mathbf{u}+\gamma\mathbf{a}) \to \gamma(c*0,0*\mathbf{u}+\gamma*\mathbf{0}) = \gamma(0,\mathbf{0}) = (0,\mathbf{0}) = \mathbf{A}_{init}$
The 4-Zero $(0,\mathbf{0})$ is still a 4-Zero in any and all inertial reference frames. A Lorentz transform on 4-Zero remains 4-Zero.

$\mathbf{A} = d\mathbf{U}/d\tau : d\mathbf{U} = \mathbf{A}\, d\tau$
$\mathbf{U} = \int \mathbf{A}\, d\tau \to \int (0,\mathbf{0})\, d\tau = (0,\mathbf{0})\tau + \mathbf{U}_{init} = \mathbf{U}_{init}$

The 4-Velocity is thus just a constant for inertial motion, the initial 4-Velocity $\mathbf{U}_{init}$.
We can choose the regular representation of 4-Velocity where the initial 3-velocity $\mathbf{u}_{init}$ is a constant to be determined
4-Velocity $\mathbf{U} = U^\mu = \gamma(c,\mathbf{u}) \to \gamma(c,\mathbf{u}_{init}) = \mathbf{U}_{init}$

$\mathbf{U} = d\mathbf{R}/d\tau : d\mathbf{R} = \mathbf{U}\, d\tau$
$\mathbf{R} = \int \mathbf{U}\, d\tau \to \int \gamma(c,\mathbf{u}_{init})\, d\tau = \int(c,\mathbf{u}_{init})\, \gamma d\tau = \int (c,\mathbf{u}_{init})\, dt = (ct,\mathbf{r} = \mathbf{u}_{init}\, t) + \mathbf{R}_{init} = (ct,\mathbf{r} = \mathbf{u}_{init}\, t) + (ct_{init},\mathbf{r}_{init})$

So, while the physical dynamic equations are general: $\mathbf{A} = d\mathbf{U}/d\tau$ and $\mathbf{U} = d\mathbf{R}/d\tau$, composed of two ProperTime Derivatives,
a free particle still has 7 unknown constants to be found $\mathbf{U}_{init}$:(3) and $\mathbf{R}_{init}$:(1+3=4)
The final equations of motion for 4D Linear Motion are:
$\mathbf{R} = (ct + ct_{init},\mathbf{r} = \mathbf{u}_{init}t + \mathbf{r}_{init})$: 4 constants t_{init} and $\mathbf{r}_{init}$
$\mathbf{U} = \gamma_{init}(c,\mathbf{u} = \mathbf{u}_{init})$: 3 constants $\mathbf{u}_{init}$
$\mathbf{A} = (0,\mathbf{a} = \mathbf{a}_{init} = \mathbf{0})$: 3 constants $\mathbf{a}_{init}$ [7 DoF's t,$\mathbf{r}$,$\mathbf{u}$] + [3 constraints $\mathbf{a}$=$\mathbf{0}$] = (10) = [10 constants t_{init},$\mathbf{r}_{init}$,$\mathbf{u}_{init}$,$\mathbf{a}_{init}$]

==

$\mathbf{U} = U^\mu = \gamma(c,\mathbf{u})$
$\mathbf{A} = A^\mu = \gamma^2(\gamma^2(\mathbf{a}\cdot\mathbf{u})/c,\gamma^2(\mathbf{a}\cdot\mathbf{u})\mathbf{u}/c^2+\mathbf{a})$

$\mathbf{U}\cdot\mathbf{U} = \gamma(c,\mathbf{u})\cdot\gamma(c,\mathbf{u}) = \gamma^2(c^2 - \mathbf{u}\cdot\mathbf{u}) = \gamma^2c^2(1 - \mathbf{u}\cdot\mathbf{u}/c^2) = c^2$

$\mathbf{U}\cdot\mathbf{A} = \gamma(c,\mathbf{u})\cdot\gamma^2(\gamma^2(\mathbf{a}\cdot\mathbf{u})/c,\gamma^2(\mathbf{a}\cdot\mathbf{u})\mathbf{u}/c^2+\mathbf{a}) = \gamma^3(\gamma^2(\mathbf{a}\cdot\mathbf{u})-\gamma^2(\mathbf{a}\cdot\mathbf{u})\mathbf{u}\cdot\mathbf{u}/c^2-\mathbf{a}\cdot\mathbf{u}) = (\mathbf{a}\cdot\mathbf{u})\gamma^3(\gamma^2-\gamma^2\mathbf{u}\cdot\mathbf{u}/c^2-1) = (\mathbf{a}\cdot\mathbf{u})\gamma^3(\gamma^2-\gamma^2\beta^2-1) = (\mathbf{a}\cdot\mathbf{u})\gamma^3(1-1) = 0$
More easily, $d/d\tau(\mathbf{U}\cdot\mathbf{U}) = 2(\mathbf{U}\cdot\mathbf{A}) = d/d\tau(c^2) = 0$, so $\mathbf{U}\cdot\mathbf{A} = 0 \leftrightarrow \mathbf{U}\perp\mathbf{A}$

Physics of Independent Parameters – Connection between Particle Dynamics:Kinematics and Angular/Linear Momentum Conservation:

Particle Dynamics using 4-Vectors = 4D (1,0)-Tensors:
4-Position $R^\mu = (ct,\mathbf{r})$ has (4) independent parameters.
4-Velocity $U^\mu = \gamma(c,\mathbf{u})$ has (3) independent parameters, due to invariant $U^\mu \eta_{\mu\nu} U^\nu = \mathbf{U}\cdot\mathbf{U} = c^2$
4-Acceleration $A^\mu = \gamma(c\gamma',\gamma'\mathbf{u}+\gamma\mathbf{a})$ has (3) independent parameters, due to invariant $U^\mu \eta_{\mu\nu} A^\nu = \mathbf{U}\cdot\mathbf{A} = 0 \leftrightarrow \mathbf{U}\perp\mathbf{A}$
Thus,
(4) + (3) + (3) = (10) independent parameters.

Can we link these to the conservation laws for a particle?
We can decompose into linear [→] and angular [↻] parts.
4-Position R^μ (4) → directly to 4-Position R^μ (4)
4-Velocity U^μ (3) + RestMass m_o (1) → to 4-LinearMomentum $P_{linear}{}^\mu = m_o U_{linear}{}^\mu$ (4)
4-Acceleration A^μ has (3) independent parameters, due to invariant $U_{rotational}{}^\mu \cdot A_{rotational}{}^\nu = 0$
So, we have (4)+(4)+(3) = (11)
However, apply the constraint $\{-R^\mu \cdot A^\nu = U_{rotational}{}^\mu \cdot U_{rotational}{}^\nu = \text{constant}\}$ for hyperbolic motion, or $-r^j \cdot a^k = u^j \cdot u^k$ for circular motion
Note the familiar rotational acceleration $|\mathbf{a}|=|\mathbf{u}^2|/|\mathbf{r}|$
1) Take the RestMass m_o used in the linear part to give hyperbolic $-(m_o)^2 R^\mu \cdot A^\nu = (m_o)U_{rotational}{}^\mu \cdot (m_o)U_{rotational}{}^\nu = P_{rotational}{}^\mu \cdot P_{rotational}{}^\nu$
or
2) Take the RestMass m_o used in the linear part to give circular $-(m_o)^2 r^j \cdot a^k = (m_o)u_{rotational}{}^j \cdot (m_o)u_{rotational}{}^k = p_{rotational}{}^j \cdot p_{rotational}{}^k$
The constraint either way lowers the system back down to (10) independent parameters.

So,
$R^\mu = R^\mu$: (4)
$P_{linear}{}^\mu = m_o U_{linear}{}^\mu$: (4) because (1) RestMass + (3) 4-Velocity
$P_{rotational}{}^\mu \cdot P_{rotational}{}^\nu = -(m_o)^2 R^\mu \cdot A^\nu$: (2) =because (3) 4-Acceleration – (1) Hyperbolic/Circular Acceleration Constraint
(4) + (4) + (2) = (10) independent parameters.

Note the $R^\mu \wedge P_{rotational}{}^\nu$ has (4) + (2) = (6) for angular [↻]
and $P_{linear}{}^\mu$ has (4) for linear [→]
(6) + (4) = (10) total independent parameters.

==

There are also the (10) parameters of the Parameterized Post-Newtonian (PPN) formalism, [from Wikipedia] which is used as a tool to compare Newtonian and Einsteinian gravity in the limit in which the gravitational field is weak and generated by objects moving slowly compared to the speed of light. In general, PPN formalism can be applied to all metric theories of gravitation in which all bodies satisfy the Einstein Equivalence Principle (EEP). The speed of light remains constant in PPN formalism and it assumes that the metric tensor is always symmetric.

1	γ	How much space curvature g_{ij} is produced by unit rest mass?
2	β	How much non-linearity is there in the superposition law for gravity g_{00}?
3	β_1	How much gravity is produced by unit kinetic energy $(1/2)\rho_0 v^2$?
4	β_2	How much gravity is produced by unit gravitational potential energy ρ_0/U?
5	β_3	How much gravity is produced by unit internal energy $\rho_0\Pi$?
6	β_4	How much gravity is produced by unit pressure p?
7	ζ	Difference between radial and transverse kinetic energy on gravity
8	η	Difference between radial and transverse stress on gravity
9	Δ_1	How much dragging of inertial frames g_{0j} . is produced by unit momentum $\rho_0 v$?
10	Δ_2	Difference between radial and transverse momentum on dragging of inertial frames

$g_{\mu\nu}$ is the 4×4 symmetric GR metric tensor (may be curved) with Greek indices μ & ν going from {0..3}.
An index of 0 will indicate the temporal direction and Latin indices i & j going from {1..3} as g_{ij} will indicate spatial directions.

In Einstein's theory, the values of these parameters are chosen:
(i) to fit Newton's Law of gravity in the limit of velocities and mass approaching zero
(ii) to ensure conservation of energy, mass, momentum, and angular momentum
(iii) to make the equations independent of the reference frame
In this notation, general relativity (GR) has PPN parameters $\gamma = \beta = \beta_1 = \beta_2 = \beta_3 = \beta_4 = \Delta_1 = \Delta_2 = 1$. and $\zeta = \eta = 0$.

These requirements are really just restatements of the other **Physics Top Ten** categories.

==

Years ago I wrote some haiku about numbers, and it seems appropriate to show them here:

Haiku of the Mystic Numbers

Zero, the center, Symbol of nothing, yet more, Granter of balance. ***One, the first to count, Unit by which all measure, It will describe truth.*** ***Two, the yin and yang, Members of duets and duels, Computer logic.*** ***Three, supports aligned, Significant for structure, Rigid triangle.*** ***Four, the dimension, Of space and time united, Relativity.*** ***Five, the sign of Man, Digits or tally marks grouped, The rays of a star.*** ***Six, magic value, Carbon chemistry of life, First perfect number.*** ***Seven, humans play, Musical notes, A through G, Luck in pair o' dice.*** ***Eight, serious math, Octonions are the last, to follow nice rules.*** ***Nine, the last digit, Days and nights on Yggdrasil, The count of all realms.*** ***Ten, most mystical, Poincaré, conservation, Fearful symmetry.***	**...** ***Infinity, end, The universal wholeness, Uncountable all.*** **...** ***i, imaginary, Complex number discovered, Elegant technique.*** ***π, the circle's sign, Perfection equidistant, Endless decimal.*** ***e, natural growth,*** $e^{2 \pi i} = 1$, ***Math: Truth is Beauty.*** ***JBW 2003-Apr,2022-Dec***

All of these (10)'s come from a 4D SpaceTime. Is there further justification for our SpaceTime being 4D?

Already, the fact that we observe all of the (10)'s in our physical measurements is evidence for 4D SpaceTime.
But, consider further...

What are the minimum number of scalars, vectors, tensors (in whatever dimension N) required to support life/observers?

If there were only time, we would be bored… ;)
If there were only relative positions, then there would be stuff, but the universe would be totally static and unchanging.
If there were positions and velocities, then there is motion, but no interaction. Just a gas of linearly-moving, non-interacting particles.
If there were positions, velocities, and accelerations, then things can speed up, slow down, turn, attract, repel, interact!
Add in a scalar variable called time and you can watch the dance unfold… Note: initially positions, then two time derivatives.
===

Consider the following:

A symmetric 2-index tensor is capable of describing covariant (frame-invariant) physics. This is a basis of General Relativity (GR).
In general, there are $(N^2+N)/2$ independent components for a symmetric 2-index tensor in a spacetime of dimension N.

If we assume as an axiom that there must be {**position r**, **velocity v**, **acceleration a**} (N-1)_dimensional **vectors**
+ a single { time t } (1)_ dimensional scalar to describe the dynamics of a system, for whatever dimension N we might be in,
then there are 3(N–1)+1 independent dynamical components.
Each spatial vector is of dimension (N–1). Time is assumed to be of just dimension (1), as this is what we observe in every experiment.
This is also related to the Ostrogradsky instability, which gives a reason why dynamics typically just uses two time derivatives along a single time dimension.
R : R →(d/dτ)→**U : U** →(d/dτ)→ **A**

Solve:
$(N^2+N)/2 = 3(N-1)+1$
$N^2/2+N/2-3N+2=0$
$N^2-5N+4=0$
$(N-4)(N-1)=0 \;\therefore\; N = 1 \text{ or } 4$

N = 1 is the case in which there is only time, no space: the divine "void" of all mythologies...
N = 4 is our 4D spacetime, with 1D time + 3D space

All other spacetime dimensions N are either (over-determined N=2,3) or (under-determined N>4):
N=1: there is only 1-time t = 1, for $(1^2+1)/2 = 1$ tensor slot
N=2:there are 1-position (r^x), 1-velocity (v^x), 1-acceleration (a^x), 1-time t = 4, for $(2^2+2)/2 = 3$ tensor slots
N=3:there are 2-position (r^x,r^y), 2-velocity (v^x,v^y), 2-acceleration (a^x,a^y), 1-time t = 7, for $(3^2+3)/2 = 6$ tensor slots
N=4:there are 3-position (r^x,r^y,r^z), 3-velocity (v^x,v^y,v^z), 3-acceleration (a^x,a^y,a^z), 1-time t = (10), for $(4^2+4)/2$ = (10) tensor slots
N=5:there are 4-position (r^w,r^x,r^y,r^z), 4-velocity (v^w,v^x,v^y,v^z), 4-acceleration (a^w,a^x,a^y,a^z), 1-time t = 13, for $(5^2+5)/2 = 15$ tensor slots
N=6:there are 5-position (r^v,r^w,r^x,r^y,r^z), 5-velocity (v^v,v^w,v^x,v^y,v^z), 5-acceleration (a^v,a^w,a^x,a^y,a^z), 1-time t = 16, for $(6^2+6)/2 = 21$ tensor slots

the counts continue growing away from each other as N increases...

Observation and experiment show that we inhabit the 4D <Time·Space>. We empirically observe 1 time and 3 space dimensions.
==

All of these (10)'s come from a 4D SpaceTime. Is there further justification for our SpaceTime being 4D?
Consider: An anti-symmetric 2-index tensor is responsible for giving covariant (frame-invariant) spatial rotational physics.
AngularMomentum is such a rotational, anti-symmetric 2-index tensor. The diagonal components are all zero.
There are $(N^2–N)/2$ independent components of an anti-symmetric 2-index tensor for a spacetime of dimension N.
(N–1) of these components are mixed Temporal (the top row of the tensor), in which each spatial dimension is singly paired with time.
There are only $(N^2–N)/2 – (N–1)$ Spatial-only tensor components (think of these as connections between spatial axes, ex. xy,xz,yz).
We want these to match the (N–1) possible Spatial dimensions (think of these as spatial axes about which they rotate, ex. z, y, x).
Why? Because that is what we observe in our universe. We see spatial rotations in a plane as being about an axis normal to the plane.

Solve:
N(N–1)/2 – (N–1) = (N–1)
N(N–1)/2 = 2(N–1)
N(N–1) = 4(N–1)
(N–4)(N–1) = 0
Again, we get the cases of: N=1 {just time} and N=4, {the 4D Spacetime that we see, with 1D time + 3D space }

The AngularMomentum [↻] 2-index AntiSymmetric Tensor has N(N–1)/2 independent components, and gives rotational motion.
The LinearMomentum [→] 1-index Tensor=Vector has N components, and gives linear motion.
N(N–1)/2 + N = N(N+1)/2 components, which is back to the number of the symmetric 2-index tensor.

Showing the actual dimensional calculations:

N=1, (time on a 0D-spatial point), there is only time again:
[0]

N=2, (time on a 1D-spatial line), there is no possibility of spatial-only rotation, only a mass moment:
[0 n^{tx}]
[-n^{tx} 0]

N=3, (time on a 2D-spatial plane), the rotation is about a direction:axis that doesn't have a dimension:
[0 n^{tx} n^{ty}]
[-n^{tx} 0 l^{xy}]
[-n^{ty} -l^{xy} 0]
The 1 spatial-only tensor doesn't fit a 2D-spatial vector. It rotates, but not about a dimensional axis.

N=4, (time on a 3D-spatial volume), the rotation is possible in all the possible spatial dimensions:
[0 n^{tx} n^{ty} n^{tz}]
[-n^{tx} 0 l^{xy} l^{xz}]
[-n^{ty} -l^{xy} 0 l^{yz}]
[-n^{tz} -l^{xz} -l^{yz} 0]
with l^{xy} = rotatation_about_z, l^{xz} = rotatation_about_y, l^{yz} = rotatation_about_z. It has # of axes = # of connections.

N=5, (time on a 4D-spatial hypervolume), then you have more tensor slots than rotational dimensions:
[0 n^{tx} n^{ty} n^{tz} n^{tw}]
[-n^{tx} 0 l^{xy} l^{xz} l^{xw}]
[-n^{ty} -l^{xy} 0 l^{yz} l^{yw}]
[-n^{tz} -l^{xz} -l^{yz} 0 l^{zw}]
[-n^{tw} -l^{xw} -l^{yw} -l^{zw} 0]
In other words, does l^{xy} = rotate_about_z, or rotate_about_w, or something else. Also, there is no way to alot the spatial-spatial tensor components isotropically to 4D-spatial vectors. In any case, it gives something that we don't observe.

N=6, (time on a 5D-spatial hypervolume), then you have more tensor slots than rotational dimensions:
[0 n^{tx} n^{ty} n^{tz} n^{tw} n^{tv}]
[-n^{tx} 0 l^{xy} l^{xz} l^{xw} l^{xv}]
[-n^{ty} -l^{xy} 0 l^{yz} l^{yw} l^{yv}]
[-n^{tz} -l^{xz} -l^{yz} 0 l^{zw} l^{zv}]
[-n^{tw} -l^{xw} -l^{yw} -l^{zw} 0 l^{wv}]
[-n^{tv} - l^{xv} - l^{yv} - l^{zv} -l^{wv} 0]
While you can divide the remaining purely spatial components into two 5D-spatial vectors, there is no symmetric way to do it.

The N-dimensional fluidic 2-index symmetric tensor $S^{\mu\nu}$ has N(N+1)/2 independent components. i.e. Stress-Energy Tensor $T^{\mu\nu}$
There is a way to use this to represent a "single-particle" fluid.

It can be decomposed into a particulate form using:
{angular} 2-index anti-symmetic tensor $A^{\mu\nu}$ with N(N–1)/2 independent components, i.e. N-AngularMomentum [↻]
+
{linear} 1-index tensor V^{μ} with N independent components, i.e. N-LinearMomentum [→]

$[N(N-1)/2]+[N] = N^2/2-N/2+N = N^2/2+N/2 = N(N+1)/2$ total independent components (anti-symmetric angular + linear=symmetric)

The {angular} 2-index anti-symmetic tensor N(N–1)/2 independent components can itself be constructed from:
two 1-index tensors $\{X^{\mu},P^{\nu}\}$ of N dimensional components each, giving 2N, using $A^{\mu\nu} = X^{\mu} \wedge P^{\nu} = X^{\mu}P^{\nu} - X^{\nu}P^{\mu}$
But the anti-symmetric tensor made thusly has 2 tensor invariants in our observable universe, lowering the # to 2N–2 independent components.
$N(N-1)/2 = 2N-2$
$N^2-N = 4N-4$
$N^2-5N+4 = 0$
$(N-4)(N-1) = 0$
$N = \{1,4\}$
==

So, we have three 1-index tensors (2 for angular part, 1 for linear part) required to equate with the original 2-index symmetric tensor.
Normally there would be 3N total independent N-vector components. But, there is the possibility of some # of tensor constraints C.

We want:
3N - # of constraints C = N(N+1)/2 independent components of symmetric 2-index tensor.
$3N - C = N(N+1)/2 = N^2/2 + N/2$
$6N - 2C = N^2 + N$
$N^2 - 5N + 2C = 0$
$N = (5\pm\sqrt{[25-4*2C]})/2 = (5\pm\sqrt{[25-8C]})/2$

if C=0, $N = (5\pm\sqrt{[25]})/2 = (5\pm5)/2 = \{0,5\}$
if C=1, $N = (5\pm\sqrt{[17]})/2$ = non-integer
if C=2, $N = (5\pm\sqrt{[9]})/2 = (5\pm3)/2 = \{1,4\}$
if C=3, $N = (5\pm\sqrt{[1]})/2 = (5\pm1)/2 = \{2,3\}$
if C>=4, N = complex

One of the 1-index tensors must be the N-Position $\mathbf{R} = (ct,r^0,\ldots,r^{N-1}) = (ct,\mathbf{r})$

Take a scalar invariant derivative (d/dτ), providing the N-Velocity.
N-Velocity $\mathbf{U} = (d/d\tau)\mathbf{R} = (d/d\tau)[(ct,r^0,\ldots r^{N-1})] = (dt/d\tau)(d/dt)[(ct,r^0,\ldots r^{N-1})] = (dt/d\tau)(c,u^0,\ldots,u^{N-1}) = \gamma(c,u^0,\ldots,u^{N-1}) = \gamma(c,\mathbf{u})$
$\mathbf{U}\cdot\mathbf{U}$ = invariant $= \gamma(c,\mathbf{u})\cdot\gamma(c,\mathbf{u}) = \gamma^2(c^2 - \mathbf{u}\cdot\mathbf{u})$
In a rest frame, $(\mathbf{u}_{rest} \to \mathbf{0})$ and $(t\to\tau)$ and $\gamma = (dt/d\tau) \to \gamma_{rest} = 1$
$\mathbf{U}\cdot\mathbf{U}$ = invariant = c^2
So, there is at least 1 constraint regardless of # of dimensions N.

Take another scalar invariant derivative (d/dτ), providing the N-Acceleration.
N-Acceleration $\mathbf{A} = (d/d\tau)\mathbf{U}$
$d[\mathbf{U}\cdot\mathbf{U}] = d[c^2] = 0 = \mathbf{A}\cdot\mathbf{U}+\mathbf{U}\cdot\mathbf{A} = 2(\mathbf{A}\cdot\mathbf{U})$, hence $(\mathbf{A}\cdot\mathbf{U}) = 0 \leftrightarrow \mathbf{U}\perp\mathbf{A}$ is another constraint.
So, there are at least 2 constraints.
This also matches the idea of **U** tangent-to-the-worldline and **A** normal-to-the-worldline. $(\mathbf{A}\cdot\mathbf{U}) = 0$ implies $(\mathbf{U}\perp\mathbf{A})$.
If only these two constraints, which we already saw for the angular, and no constraints on the linear part, then again {1,4}.

Based on observation, **R** + **U** + **A** is sufficient to describe physical dynamics in a 4D spacetime.

So, this derivation implies that, based on tensor mathematics, a physical system can be described with:
a) one symmetric 2-index tensor (typically fluid density but including possibility of a "one-particle" fluid)
b) one (angular [↻]) anti-symmetric 2-index tensor + one (linear [→]) 1-index tensor: Conservation of Angular/Linear Momentum
c) three 1-index tensors, using just two ProperTime derivatives (see Ostrogradsky's Instability): **R**, $\mathbf{U}=d\mathbf{R}/d\tau$, $\mathbf{A}=d^2\mathbf{R}/d\tau^2$

All of these have (10) independent components in a 4D spacetime.
10 for the Symmetric Tensor
6+4 = 10 for the Angular Anti-Symmetric 4-Tensor + Linear 4-Vector
4+3+3 = 10 for the three 4-Vectors, which are linked by two time derivatives, matching Ostrogradsky.

Another way to consider the tensor-vector arguments:

For any spacetime 2-index tensor of dimension N, we have (1) temporal and (N-1) spatial dimensions in an N×N matrix.

$[T^{tt}, T^{tr}]$
$[T^{rt}, T^{rr}]$

T^{tt} is a 1-dimensional purely-temporal scalar
T^{tr} is a (N-1) dimensional mixed-timespace vector, with each **r** component paired with t
T^{rt} is a (N-1) dimensional mixed-timespace vector, with each **r** component paired with t
T^{rr} is a (N-1) dimensional purely-spatial tensor in an (N-1)×(N-1) matrix

Based on measurement homogeneity and isotropic arguments, we want to divide up the purely-spatial tensor T^{rr} into an integer number of (N-1) dimensional pairings.
We can pair each spatial component with the time component. ex. the top mixed row $[T^{tx}, T^{ty}, T^{tz}, ...]$
We can pair each spatial component with itself. ex. the spatial diagonal $[T^{xx}, T^{yy}, T^{zz}, ...]$
These will both always have the same number (N-1) of components.

$[T^{tt}\ T^{tx}\ T^{ty}\ T^{tz}]$
$[T^{xt}\ T^{xx}\ T^{xy}\ T^{xz}]$
$[T^{yt}\ T^{yx}\ T^{yy}\ T^{yz}]$
$[T^{zt}\ T^{zx}\ T^{zy}\ T^{zz}]$

Now, there needs to be a simple, symmetric way of dividing the remaining spatial components into (N-1) dimensional vectors.
The only symmetric choice is the N=4 choice of $[T^{xy}, T^{xz}, T^{yz}]$, which has all the remaining (N-1) components.

Any other dimensional choice N will either not have all (N-1) dimensional spatial-vectors, or will have to make a non-symmetric choice in the spatial components of the remaining (N-1) vectors.

4-Position	$R^\mu = (ct,\mathbf{r}) = \mathbf{R} \in$ <event>	[m]	$(ct,\mathbf{r}) \to (ct,x,y,z)$ only Lorentz Invariant, not Poincaré Invariant
4-Velocity	$U^\mu = \gamma(c,\mathbf{u}) = \mathbf{U} = d\mathbf{R}/d\tau$	[m/s]	Lorentz Gamma Factor $\gamma = 1/\sqrt{[1 - (u/c)^2]} = dt/d\tau$
4-Acceleration	$A^\mu = \gamma(c\gamma', \gamma'\mathbf{u}+\gamma\mathbf{a}) = \mathbf{A} = d\mathbf{U}/d\tau = d^2\mathbf{R}/d\tau^2$	[m/s²]	$A^\mu = (\gamma^4(\mathbf{a}\cdot\mathbf{u})/c, \gamma^4(\mathbf{a}\cdot\mathbf{u})\mathbf{u}/c^2+\gamma^2\mathbf{a}) = \gamma^2(\gamma^2(\mathbf{a}\cdot\mathbf{u})/c, \gamma^2(\mathbf{a}\cdot\mathbf{u})\mathbf{u}/c^2+\mathbf{a})$

$\gamma = 1/\sqrt{[1 - \beta^2]}$
$\gamma^2 = 1/(1 - \beta^2)$
$\gamma^2(1 - \beta^2) = 1$
$\gamma^2 - \gamma^2\beta^2 = 1$

$\mathbf{U}\cdot\mathbf{U} = \gamma(c,\mathbf{u})\cdot\gamma(c,\mathbf{u}) = \gamma^2(c^2 - \mathbf{u}\cdot\mathbf{u}) = \gamma^2c^2(1 - \mathbf{u}\cdot\mathbf{u}/c^2) = c^2$

$\mathbf{U}\cdot\mathbf{A} = \gamma(c,\mathbf{u})\cdot\gamma^2(\gamma^2(\mathbf{a}\cdot\mathbf{u})/c, \gamma^2(\mathbf{a}\cdot\mathbf{u})\mathbf{u}/c^2+\mathbf{a}) = \gamma^3(\gamma^2(\mathbf{a}\cdot\mathbf{u})-\gamma^2(\mathbf{a}\cdot\mathbf{u})\mathbf{u}\cdot\mathbf{u}/c^2-\mathbf{a}\cdot\mathbf{u}) = (\mathbf{a}\cdot\mathbf{u})\gamma^3(\gamma^2-\gamma^2\mathbf{u}\cdot\mathbf{u}/c^2-1) = (\mathbf{a}\cdot\mathbf{u})\gamma^3(\gamma^2-\gamma^2\beta^2-1) = (\mathbf{a}\cdot\mathbf{u})\gamma^3(1-1) = 0$
More easily, $d/d\tau(\mathbf{U}\cdot\mathbf{U}) = 2(\mathbf{U}\cdot\mathbf{A}) = d/d\tau(c^2) = 0$, so $\mathbf{U}\cdot\mathbf{A} = 0$ $\mathbf{U}\perp\mathbf{A}$

Why only 2 orders of derivatives in the SpaceTime dimension derivation? See **Ostrogradsky instability**.
3-position = $\mathbf{x} = d^0\mathbf{x}/dt^0$
3-velocity = $\mathbf{v} = d^1\mathbf{x}/dt^1$
3-acceleration $\mathbf{a} = d^2\mathbf{x}/dt^2$
The process is about getting the # of independent parameters. Certainly, we couldn't do physics without positions or velocities, nor without time itself. Adding in the one more level gives accelerations, which allows:
{ interactions, speeding up, slowing down, attraction, repulsion, forces, potentials, ... Note the correct way to do an anthropomorphic argument.}

That empirically sounds like the universe that we observe. That this gives an answer where # of dimensions N = 1 or 4 is quite interesting to me.
Remember, these arguments don't specify the # of dimensions at the start. N = {1 or 4} is the **result**.
In other words, not only is the universe/spacetime 4D, but it has to be either 1D or 4D based on these analyses.

What about $\mathbf{A}\cdot\mathbf{A} = -\alpha^2$: No extra constraint, because there is extra parameter α.

$d/d\tau(\mathbf{U}\cdot\mathbf{A}) = (\mathbf{A}\cdot\mathbf{A})+(\mathbf{U}\cdot\mathbf{J}) = d/d\tau(0) = 0$, so $\mathbf{U}\cdot\mathbf{J} = -\mathbf{A}\cdot\mathbf{A} = \alpha^2$, and then $\mathbf{J} = (\alpha/c)\mathbf{U}$, meaning the 4-Jerk is ~ to 4-Velocity, using the same parameter α from the 4-Acceleration. Also, $\mathbf{J}\cdot\mathbf{A} = (\alpha^2/c^2)\mathbf{U}\cdot\mathbf{A} = (\alpha^2/c^2)0 = 0$. No new information.

$d/d\tau(\mathbf{A}\cdot\mathbf{J}) = (\mathbf{J}\cdot\mathbf{J})+(\mathbf{A}\cdot\mathbf{S}) = d/d\tau(0) = 0$, so $\mathbf{A}\cdot\mathbf{S} = -\mathbf{J}\cdot\mathbf{J} = -(\alpha^2/c^2)^2(\mathbf{U}\cdot\mathbf{U}) = -\alpha^4/c^2$, and then $\mathbf{S} = (\alpha/c)\mathbf{A}$, meaning the 4-Snap is ~ to 4-Acceleration, again using the same parameter α from the 4-Acceleration. Also, $\mathbf{S}\cdot\mathbf{J} = 0$.
Continue this downward ladder of derivatives. Each is 4D Orthogonal to the last, and proportional to the next to last.

Analytic Mechanics & Ostrogradsky Instability: https://en.wikipedia.org/wiki/Ostrogradsky_instability
Basically, it is a prescription for showing that physics typically uses at most two time derivatives (d/dt) to describe dynamic systems.
I got to thinking about the form of equations of motion using different techniques.

The usual SR 4-Vectors and 4-Scalars:

4-{Position}Gradient	$\partial = \partial^\mu = (\partial_t/c,-\nabla) = \partial_\mathbf{r} = \partial/\partial R_\mu$	$\partial_\mathbf{r}[\tau] = \mathbf{U}/c^2$
4-{Velocity}Gradient	$(\partial_{u\_t}/c,-\nabla_\mathbf{u}) = \partial_\mathbf{u} = \partial/\partial U_\mu$	$\partial_\mathbf{u}[\tau] = \mathbf{R}/c^2$
4-Position	$\mathbf{R} = R^\mu = (ct,\mathbf{r})$	
4-Velocity	$\mathbf{U} = U^\mu = \gamma(c,\mathbf{u}) = (d/d\tau)\mathbf{R}$	$(\mathbf{U}\cdot\mathbf{R})=c^2\tau$ $(\mathbf{U}\cdot\partial_\mathbf{r})[\tau]=1$
4-Acceleration	$\mathbf{A} = A^\mu = \gamma(c\gamma',\gamma'\mathbf{u}+\gamma\mathbf{a}) = (d/d\tau)\mathbf{U}$	$\mathbf{A} = (d/d\tau)\mathbf{U} = (d^2/d\tau^2)\mathbf{R} = (d/d\tau)(d/d\tau)\mathbf{R} = (\mathbf{U}\cdot\partial)(\mathbf{U}\cdot\partial)\mathbf{R}$
4-Momentum	$\mathbf{P} = P^\mu = (E/c=mc,\mathbf{p}=m\mathbf{u}) = m_o\mathbf{U}$	
4-VectorPotential	$\mathbf{A_{EM}} = A_{EM}{}^\mu = (\varphi/c,\mathbf{a_{EM}})$	
4-TotalMomentum	$\mathbf{P_T} = P_T{}^\mu = (H/c,\mathbf{p_T}) = \mathbf{P}+q\mathbf{A_{EM}}$	Includes effects of particle in EM potential $\mathbf{Q} = q\mathbf{A_{EM}}$
ProperTime Derivative	$(d/d\tau) = (\mathbf{U}\cdot\partial) = (\mathbf{U}\cdot\partial_\mathbf{r}) = (\mathbf{A}\cdot\partial_\mathbf{u})$	

Vector based: $\mathbf{U} = (d/d\tau)\mathbf{R}$ Relativistic ProperTime Derivative Kinematics
Function based: $\partial_\mathbf{r} = (d/d\tau)\partial_\mathbf{u}$ Relativistic Euler-Lagrange
==

- Kinematics, time-position-velocity-acceleration {t,$\mathbf{r}$,$\mathbf{u}$,$\mathbf{a}$}-based, using $\partial_\mathbf{r}$ & $\partial_\mathbf{u}$ or $\partial_\mathbf{r}$ twice:

$\mathbf{A} = (d/d\tau)\mathbf{U} = (d^2/d\tau^2)\mathbf{R} = (d/d\tau)(d/d\tau)\mathbf{R} = (\mathbf{U}\cdot\partial_\mathbf{r})(\mathbf{U}\cdot\partial_\mathbf{r})\mathbf{R} = (\mathbf{A}\cdot\partial_\mathbf{u})(\mathbf{U}\cdot\partial_\mathbf{r})\mathbf{R} = (\mathbf{A}\cdot\partial_\mathbf{u})\mathbf{U} = \mathbf{A}$

- Euler-Lagrange-based, using $\partial_\mathbf{r}$ & $\partial_\mathbf{u}$:

$\partial_\mathbf{r} = (d/d\tau)\partial_\mathbf{u} = (\mathbf{U}\cdot\partial_\mathbf{r})\partial_\mathbf{u}$ $\partial_\mathbf{r}(\mathbf{U}\cdot\mathbf{R})=(d/d\tau)\partial_\mathbf{u}(\mathbf{U}\cdot\mathbf{R})$ $\partial_\mathbf{r}(c^2\tau)=(d/d\tau)\partial_\mathbf{u}(c^2\tau)$
$\mathbf{U} = (d/d\tau)\mathbf{R}$ $\mathbf{U} = (d/d\tau)\mathbf{R}$

- d'Alembertian wave-based, using $\partial_\mathbf{r}$ twice:

$\partial\cdot\partial = \partial_\mathbf{r}\cdot\partial_\mathbf{r}$ = Constant Invariant $\psi = \psi_o e\string^(iS/\hbar)$

- Hamiltonian-based, using $\partial_\mathbf{r}$ & $\partial_\mathbf{PT}$:

$(d/d\tau)[R^\alpha] = (\mathbf{U}\cdot\partial_\mathbf{r})[\mathbf{R}] = (\partial_\mathbf{PT})[H_o] = \mathbf{U}$ $H+L= \mathbf{p_T}\cdot\mathbf{u}$
$(d/d\tau)[P_T{}^\alpha] = (\mathbf{U}\cdot\partial_\mathbf{r})[\mathbf{P_T}] = (\partial_\mathbf{r})[H_o]$ $H = \gamma(\mathbf{P_T}\cdot\mathbf{U})$ $H_o = (\mathbf{P_T}\cdot\mathbf{U})$ $L= -(\mathbf{P_T}\cdot\mathbf{U})/\gamma$
$(\mathbf{U}\cdot\partial_\mathbf{r})[\mathbf{P_T}] = (\mathbf{U}\cdot\partial_\mathbf{r})(H/c,\mathbf{p_T}) = \mathbf{U}\cdot(\partial_\mathbf{r}H/c,\partial_\mathbf{r}\mathbf{p_T}) = (\partial_\mathbf{r})[H_o]$

- Hamilton-Jacobi-based, using $\partial_\mathbf{r}$ & $\partial_\mathbf{PT}$:

$\mathbf{P_T} = -\partial_\mathbf{r}[S] = (H/c,\mathbf{p_T}) = -(\partial_t/c,-\nabla)[S] = (-\partial_t/c,\nabla)[S]$ $S = -\int(\mathbf{P_T}\cdot d\mathbf{R}) \to -(\mathbf{P_T}\cdot\mathbf{R})$+const if $\mathbf{P_T}$ not a function of $\mathbf{R}$
$\mathbf{R} = -\partial_\mathbf{PT}[S] = (ct,\mathbf{r})$ $S = \int(L\ dt)$ + const
==
Note that all of these have two 4-Gradients ∂ involved, with respect to { Position $\mathbf{R}$ or Velocity $\mathbf{U}$ or TotalMomentum $\mathbf{P_T}$ }

These 4-Gradient "open forms" assume that there exist solutions:
{ $\mathbf{A} = (\mathbf{U}\cdot\partial_\mathbf{r})(\mathbf{U}\cdot\partial_\mathbf{r})[\mathbf{R}]$ = Kinematic Eqn, with {$\mathbf{A}$,$\mathbf{R}$} as the Kinematic Vector solution }
{ $\partial_\mathbf{r}[L] = (d/d\tau)\partial_\mathbf{u}[L]$ = Euler-Lagrange Eqn, with L or L_o as the Lagrangian solution } $L= -(\mathbf{P_T}\cdot\mathbf{U})/\gamma$ $L_o = -(\mathbf{P_T}\cdot\mathbf{U})$
{ $(\partial_\mathbf{r}\cdot\partial_\mathbf{r})[\psi] = \text{const}[\psi]$ = d'Alembertian Wave Eqn. with ψ as the Wave solution }
{ $(\mathbf{U}\cdot\partial_\mathbf{r})[\mathbf{P_T}] = (\partial_\mathbf{r})[H_o]$ = Hamiltonian Eqn. with {$\mathbf{P_T}$, H_o} as the Hamiltonian solution } $(\mathbf{U}\cdot\partial_\mathbf{r})[\mathbf{P_T}] = (\partial_\mathbf{r})[\mathbf{U}\cdot\mathbf{P_T}]$ & $(\mathbf{U}\cdot\partial_\mathbf{r})[\mathbf{R}] = (\partial_\mathbf{PT})[\mathbf{U}\cdot\mathbf{P_T}] = \mathbf{U}$
{ $\mathbf{P_T} = -\partial_\mathbf{r}[S]$ = Hamiltonian-Jacobi Eqn. with S as the Action solution }

{ $\mathbf{U} = d\mathbf{R}/d\tau = (d/d\tau)\mathbf{R}$ }: gives a 4D tensorial dynamics of particles
{ $\partial_\mathbf{r}[L] = (d/d\tau)\partial_\mathbf{u}[L]$ = Euler-Lagrange }: gives a 4D tensorial dynamics of almost anything

$\partial\cdot\partial$ = Invariant. This is the d'Alembertian Wave Equation.
{$\partial\cdot\partial[\psi]$ = d'Alembertian wave Eqn.}: gives a 4D tensorial dynamics of waves
$d[f] = (d\mathbf{R}\cdot\partial_\mathbf{r})[f] = (dt\partial/\partial_t+dx\partial/\partial_x+dy\partial/\partial_y+dz\partial/\partial_z)[f] = (d\mathbf{U}\cdot\partial_\mathbf{u})[f]$: From rules of calculus
$d = (d\mathbf{R}\cdot\partial_\mathbf{r}) = (d\mathbf{U}\cdot\partial_\mathbf{u})$ Differential d is Lorentz Invariant
$(d/d\tau) = (\mathbf{U}\cdot\partial_\mathbf{r}) = (\mathbf{A}\cdot\partial_\mathbf{u})$ ProperTime Derivative is Lorentz Invariant

d=d
$(d\mathbf{R}\cdot\partial_\mathbf{r}) = (d\mathbf{U}\cdot\partial_\mathbf{u})$
$(d\mathbf{R}\cdot\partial_\mathbf{r}) = [d(d\mathbf{R}/d\tau)\cdot\partial_\mathbf{u}]$
$(d\mathbf{R}\cdot\partial_\mathbf{r}) = [(d)(d\mathbf{R})/d\tau)\cdot\partial_\mathbf{u}]$
$(d\mathbf{R}\cdot\partial_\mathbf{r}) = [(d\mathbf{R})(d/d\tau)\cdot\partial_\mathbf{u}]$
$(d\mathbf{R}\cdot\partial_\mathbf{r}) = [d\mathbf{R}\cdot(d/d\tau)\partial_\mathbf{u}]$
$(\partial_\mathbf{r}) = (d/d\tau)\partial_\mathbf{u}$: The Relativistic Euler-Lagrange Equation

$(\partial_\mathbf{r})[\mathbf{U}\cdot\mathbf{P_T}] = (\partial_\mathbf{r})[\mathbf{U}]\cdot\mathbf{P_T} + \mathbf{U}\cdot(\partial_\mathbf{r})[\mathbf{P_T}] = 0 + \mathbf{U}\cdot(\partial_\mathbf{r})[\mathbf{P_T}]$, then rearrange $(\mathbf{U}\cdot\partial_\mathbf{r})[\mathbf{P_T}] = (\partial_\mathbf{r})[\mathbf{U}\cdot\mathbf{P_T}] = (\partial_\mathbf{r})[H_o]$ The Relativistic Hamilton Eqns.
Note by inspection that: $(\mathbf{U}\cdot\partial_\mathbf{r})[\mathbf{R}] = (\partial_\mathbf{PT})[\mathbf{U}\cdot\mathbf{P_T}] = (\partial_\mathbf{PT})[H_o] = \mathbf{U}$

Relativistic Hamilton's Equations:

Simple Tensorial Derivation: Note that $H_o = (\mathbf{P_T}\cdot\mathbf{U})$

$(\partial_r)[\mathbf{U}\cdot\mathbf{P_T}] = (\partial_r)[\mathbf{U}]\cdot\mathbf{P_T} + \mathbf{U}\cdot(\partial_r)[\mathbf{P_T}] = 0 + \mathbf{U}\cdot(\partial_r)[\mathbf{P_T}]$, then rearrange $(\mathbf{U}\cdot\partial_r)[\mathbf{P_T}] = (\partial_r)[\mathbf{U}\cdot\mathbf{P_T}] = (\partial_r)[H_o] = (d/d\tau)(\mathbf{P_T})$

Note by inspection that: $(\mathbf{U}\cdot\partial_r)[\mathbf{R}] = (\partial_{\mathbf{PT}})[\mathbf{U}\cdot\mathbf{P_T}] = (\partial_{\mathbf{PT}})[H_o] = \mathbf{U}$

$(d/d\tau)[R^\alpha] = (\mathbf{U}\cdot\partial_r)[\mathbf{R}] = (\partial_{\mathbf{PT}})[H_o] = (\partial_{\mathbf{PT}})[\mathbf{U}\cdot\mathbf{P_T}] = \mathbf{U}$ $\quad H = \gamma(\mathbf{P_T}\cdot\mathbf{U}) = \gamma H_o$ $\quad H_o = (\mathbf{P_T}\cdot\mathbf{U}) = \gamma(H-\mathbf{p_T}\cdot\mathbf{u}) = H/\gamma$

$(d/d\tau)[P_T{}^\alpha] = (\mathbf{U}\cdot\partial_r)[\mathbf{P_T}] = (\partial_r)[H_o] = (\partial_r)[\mathbf{U}\cdot\mathbf{P_T}]$

4-Vector Form (Relativistic)	3-vector Form (Classical)
$(d/d\tau)[R^\alpha] = (\partial_{Pt}{}^\alpha)[H_o] = U^\alpha$	$d\mathbf{q}/dt = +\,\partial H/\partial\mathbf{p_T} = \mathbf{u}$
$(d/d\tau)[P_T{}^\alpha] = (\partial_r{}^\alpha)[H_o]$	$d\mathbf{p_T}/dt = -\,\partial H/\partial\mathbf{q}$

$H_o = (\mathbf{P_T}\cdot d\mathbf{R}/d\tau) = (\mathbf{P_T}\cdot(d/d\tau)\mathbf{R}) = (\mathbf{P_T}\cdot(\mathbf{U}\cdot\partial_r)\mathbf{R}) \rightarrow (\mathbf{U}\cdot\partial_r)(\mathbf{P_T}\cdot\mathbf{R})$ assuming $\mathbf{P_T}$ is constant

$(\partial_r)[H_o] = (\partial_r)(\mathbf{U}\cdot\partial_r)(\mathbf{P_T}\cdot\mathbf{R}) = (\mathbf{U}\cdot\partial_r)(\partial_r)(\mathbf{P_T}\cdot\mathbf{R}) = (\mathbf{U}\cdot\partial_r)(\mathbf{P_T}) = (d/d\tau)(\mathbf{P_T})$

$(\mathbf{U}\cdot\partial_r)[\mathbf{P_T}] = (\mathbf{U}\cdot\partial_r)(H/c,\mathbf{p_T}) = \mathbf{U}\cdot(\partial_r H/c,\partial_r\mathbf{p_T}) = (\partial_r)[H_o]$

$(\mathbf{U}\cdot\partial_r)[\mathbf{R}] = (d/d\tau)[R^\alpha] = \gamma(d/dt)[R^\alpha] = (\partial_{\mathbf{PT}})[H_o = (\mathbf{P_T}\cdot\mathbf{U}) = \gamma(H-\mathbf{p_T}\cdot\mathbf{u})] = \mathbf{U}$

$(\mathbf{U}\cdot\partial_r)[(r^0{=}ct,\mathbf{r})] = (d/d\tau)[(r^0{=}ct,r^k)] = \gamma(d/dt)[(r^0{=}ct,r^k)] = (c\partial/\partial H,-\nabla_{\mathbf{pT}})[H_o = (\mathbf{P_T}\cdot\mathbf{U}) = \gamma(H-\mathbf{p_T}\cdot\mathbf{u})] = (\gamma c,\gamma\mathbf{u}) = \mathbf{U}$

$(\mathbf{U}\cdot\partial_r)$ $(d/d\tau)$ $\gamma(d/dt)$	[	$\mathbf{R}$ R^α (r^0,r^k) $(ct,\mathbf{r})$	]=	$\partial_{\mathbf{PT}}$ $\partial_{\mathbf{PT}}{}^\alpha$ $(c\partial/\partial H,-\nabla_{\mathbf{pT}})$	[	H_o H/γ $(\mathbf{P_T}\cdot\mathbf{U})$ $(\mathbf{P}+q\mathbf{A})\cdot\mathbf{U}$ $(\mathbf{P}\cdot\mathbf{U})+q(\mathbf{A}\cdot\mathbf{U})$ $\gamma(H-\mathbf{p_T}\cdot\mathbf{u})$	]=	$\mathbf{U}$ U^α $\gamma(c,\mathbf{u})$ $(\gamma c,\gamma\mathbf{u})$

$\gamma d\mathbf{r}/dt = -\nabla_{\mathbf{pT}}[H_o = H/\gamma = \gamma(H-\mathbf{p_T}\cdot\mathbf{u})] = \gamma\mathbf{u}$

$d\mathbf{r}/dt = -\nabla_{\mathbf{pT}}[H/\gamma^2 = (H-\mathbf{p_T}\cdot\mathbf{u})] = \mathbf{u}$

$d\mathbf{r}/dt = \nabla_{\mathbf{pT}}[(\mathbf{p_T}\cdot\mathbf{u}-H) = -H/\gamma^2] = \mathbf{u}$

$d\mathbf{r}/dt = \nabla_{\mathbf{pT}}[(\mathbf{p_T}\cdot\mathbf{u}-H) = -H(1-\boldsymbol{\beta}\cdot\boldsymbol{\beta})] = \mathbf{u}$

$d\mathbf{r}/dt = \nabla_{\mathbf{pT}}[+2H-H(1-\boldsymbol{\beta}\cdot\boldsymbol{\beta})] = \mathbf{u}$: We can add in multiples of H without affecting MomentumGradient

$d\mathbf{r}/dt = \nabla_{\mathbf{pT}}[H(1+\boldsymbol{\beta}\cdot\boldsymbol{\beta})] = \mathbf{u}$

Take limit-case of ($|v| \ll c$) = ($|\beta| \ll 1$)

$d\mathbf{r}/dt \sim \nabla_{\mathbf{pT}}[H] = \mathbf{u}$

$\mathbf{F}+qd\mathbf{A}/d\tau$ $\mathbf{F}+q\mathbf{U}\cdot\partial_r[\mathbf{A}]$ $F^\alpha+qU_\beta\partial^\beta[A^\alpha]$	=	$(\mathbf{U}\cdot\partial_r)$ $(d/d\tau)$ $\gamma(d/dt)$	[	$\mathbf{P_T}$ $P_T{}^\alpha$ $\mathbf{P}+q\mathbf{A}$ $(H/c,\mathbf{p_T})$	]=	∂_r ∂^α $(\partial/c\partial t,-\nabla_r)$	[	H_o H/γ $(\mathbf{P_T}\cdot\mathbf{U})$ $(\mathbf{P}+q\mathbf{A})\cdot\mathbf{U}$ $(\mathbf{P}\cdot\mathbf{U})+q(\mathbf{A}\cdot\mathbf{U})$ $\gamma(H-\mathbf{p_T}\cdot\mathbf{u})$	]=	$0 + q\partial_r[\mathbf{A}]\cdot\mathbf{U}$ $0 + q\partial^\alpha[A^\beta]U_\beta$

$F^\alpha+qU_\beta\partial^\beta[A^\alpha] = 0 + q\partial^\alpha[A^\beta]U_\beta$

$F^\alpha = q(\partial^\alpha[A^\beta] - \partial^\beta[A^\alpha])U_\beta$

$F^\alpha = q\boldsymbol{F}^{\alpha\beta}U_\beta$ $\qquad$ with $\boldsymbol{F}^{\alpha\beta}$ = Faraday EM Tensor = $(\partial^\alpha[A^\beta] - \partial^\beta[A^\alpha])$

$F^i = q(\partial^i A^\beta - \partial^\beta A^i)U_\beta$

$F^i = q(\partial^i A^0 - \partial^0 A^i)U_0 + q(\partial^i A^j - \partial^j A^i)U_j$

$\gamma\mathbf{f} = q(-\nabla[\varphi/c] - (\partial_t/c)\mathbf{a})(\gamma c) + q(-\nabla[\mathbf{a}\cdot\mathbf{u}] - -\mathbf{u}\cdot\nabla[\mathbf{a}])\gamma$

$\mathbf{f} = q(-\nabla[\varphi/c] - (\partial_t/c)\mathbf{a})(c) + q(\mathbf{u}\cdot\nabla[\mathbf{a}]-\nabla[\mathbf{a}\cdot\mathbf{u}])$

$\mathbf{f} = q(-\nabla[\varphi] - \partial_t\mathbf{a} + \mathbf{u}\cdot\nabla[\mathbf{a}] - \nabla[\mathbf{a}\cdot\mathbf{u}])$

$\mathbf{f} = q(-\nabla[\varphi] - \partial_t\mathbf{a} + \mathbf{u}\times\mathbf{b})$

$\mathbf{f} = q(\mathbf{e} + \mathbf{u}\times\mathbf{b})$

Take the limit of $\{\, |\nabla[\varphi]| \gg |\partial_t\mathbf{a} - \mathbf{u}\times\mathbf{b}| \,\}$

$\mathbf{f} \sim q(-\nabla[\varphi]) = -\nabla[q\varphi] = -\nabla[V] = -$**Grad**[Potential]

$d\mathbf{p}/dt = \mathbf{f} \sim q(-\nabla[\varphi]) = -\nabla[q\varphi] = -\nabla[V] = -$**Grad**[Potential] : $d\mathbf{p}/dt = -\,\partial H/\partial\mathbf{q}$

The Classical **Force** = -**Grad**[Potential]

when $\{\, |\nabla[\varphi]| \gg |\partial_t\mathbf{a} - \mathbf{u}\times\mathbf{b}| \,\}$ or when $\{\, \mathbf{a} = \mathbf{0}$, non-magnetic$\}$

4-{Position}Gradient $\partial = \partial^\mu = (\partial_t/c, -\nabla) = \partial_R = \partial/\partial R_\mu$ $\partial_R(\tau) = \mathbf{U}/c^2$
4-{Velocity}Gradient $(\partial_{u\_t}/c, -\nabla_u) = \partial_U = \partial/\partial U_\mu$ $\partial_U(\tau) = \mathbf{R}/c^2$
4-{TotalMomentum}Gradient $(\partial_{pT\_t}/c, -\nabla_{pT}) = \partial_{PT} = \partial/\partial P_{T\mu}$
4-Position $\mathbf{R} = R^\mu = (ct, \mathbf{r})$
4-Velocity $\mathbf{U} = U^\mu = \gamma(c, \mathbf{u}) = (d/d\tau)\mathbf{R}$ $(\mathbf{U}\cdot\mathbf{R}) = c^2\tau$ $(\mathbf{U}\cdot\partial_R)[\tau] = 1$
4-VectorPotential $\mathbf{A_{EM}} = A_{EM}{}^\mu = (\varphi/c, \mathbf{a_{EM}})$
4-TotalMomentum $\mathbf{P_T} = P_T{}^\mu = (H/c, \mathbf{p_T}) = \mathbf{P} + q\mathbf{A_{EM}}$ Includes effects of particle in EM potential $\mathbf{Q} = q\mathbf{A_{EM}}$

$\mathbf{P_T} = (H_o/c^2)\mathbf{U} = (H/c, \mathbf{p_T}) = (H_o/c^2)\gamma(c, \mathbf{u}) = (\gamma H_o/c^2)(c, \mathbf{u}) = (\gamma H_o/c)(1, \boldsymbol{\beta})$
Temporal part: $H/c = (\gamma H_o/c)(1)$, or $H = \gamma H_o$
Spatial part: $\mathbf{p_T} = (\gamma H_o/c)(\boldsymbol{\beta}) = H\boldsymbol{\beta}/c = H\mathbf{u}/c^2$ and $\mathbf{p_T}\cdot\mathbf{u} = H\mathbf{u}\cdot\mathbf{u}/c^2 = H\boldsymbol{\beta}\cdot\boldsymbol{\beta} = \gamma H_o\boldsymbol{\beta}\cdot\boldsymbol{\beta} = \gamma\boldsymbol{\beta}\cdot\boldsymbol{\beta}H_o$

Rest Hamiltonian $(\mathbf{P_T}\cdot\mathbf{U}) = (H/c, \mathbf{p_T})\cdot\gamma(c, \mathbf{u}) = \gamma(H - \mathbf{p_T}\cdot\mathbf{u}) = H_o = (\mathbf{P_T}\cdot\mathbf{U}) = (H_o/c^2)\mathbf{U}\cdot\mathbf{U} = (H_o/c^2)(c^2) = H_o$

Lorentz Gamma Factor:
$\gamma = 1/\text{Sqrt}[(1-\boldsymbol{\beta}\cdot\boldsymbol{\beta})]$
$\gamma^2 = 1/[(1-\boldsymbol{\beta}\cdot\boldsymbol{\beta})]$
$(1 - \boldsymbol{\beta}\cdot\boldsymbol{\beta})\gamma^2 = 1$
$(\gamma^2 - \gamma^2\boldsymbol{\beta}\cdot\boldsymbol{\beta}) = 1$
$\gamma^2 - 1 = \gamma^2\boldsymbol{\beta}\cdot\boldsymbol{\beta}$
$\gamma - 1/\gamma = \gamma\boldsymbol{\beta}\cdot\boldsymbol{\beta}$ This remains an identity

$\gamma(\mathbf{P_T}\cdot\mathbf{U}) + -(\mathbf{P_T}\cdot\mathbf{U})/\gamma = \gamma\boldsymbol{\beta}\cdot\boldsymbol{\beta}(\mathbf{P_T}\cdot\mathbf{U})$ Apply the Lorentz Scalar $(\mathbf{P_T}\cdot\mathbf{U})$ to the identity
$\gamma(H_o) + -(H_o)/\gamma = \gamma\boldsymbol{\beta}\cdot\boldsymbol{\beta}H_o$
$\gamma(H_o) + -(H_o)/\gamma = \mathbf{p_T}\cdot\mathbf{u}$
$H + L = \mathbf{p_T}\cdot\mathbf{u}$
$H_o + L_o = 0$

Identify: $L = -(H_o)/\gamma = L_o/\gamma$ or $L_o = -H_o$
$H = \gamma(\mathbf{P_T}\cdot\mathbf{U})$ and $L = -(\mathbf{P_T}\cdot\mathbf{U})/\gamma$
$H_o = (\mathbf{P_T}\cdot\mathbf{U})$ and $L_o = -(\mathbf{P_T}\cdot\mathbf{U})$

$(\partial_R)[H_o] = (\partial_R)(\mathbf{P_T}\cdot\mathbf{U}) = (\partial_R)(\mathbf{U}\cdot\partial_R)(\mathbf{P_T}\cdot\mathbf{R}) = (\mathbf{U}\cdot\partial_R)(\partial_R)(\mathbf{P_T}\cdot\mathbf{R}) = (\mathbf{U}\cdot\partial_R)(\mathbf{P_T}) = (d/d\tau)(\mathbf{P_T})$: $(d/d\tau)(\mathbf{P_T}) = (\partial_R)[H_o]$
Hamilton's Eqns.
$(\partial_{PT})[H_o] = (\partial_{PT})(\mathbf{P_T}\cdot\mathbf{U}) = \mathbf{U} = (d/d\tau)(\mathbf{R})$: $(d/d\tau)(\mathbf{R}) = (\partial_{PT})[H_o]$

$(\partial_U)[H_o] = (\partial_U)(\mathbf{P_T}\cdot\mathbf{U}) = \mathbf{P_T}$
$(\partial_R)[H_o] = (d/d\tau)(\mathbf{P_T}) = (d/d\tau)(\partial_U)[H_o]$
$(\partial_R)[-L_o] = (d/d\tau)(\partial_U)[-L_o]$
$(\partial_R)[L_o] = (d/d\tau)(\partial_U)[L_o]$ Euler-Lagrange Eqn.

Let $L_o = (\mathbf{R}\cdot\mathbf{U})*(m_o/\tau_{Planck}) = (c^2\tau)*(m_o/\tau_{Planck}) = (m_oc^2\tau/\tau_{Planck}) = (E_o\tau/\tau_{Planck})$
Then:
$(\partial_R)[(\mathbf{R}\cdot\mathbf{U})*(m_o/\tau_{Planck})] = (d/d\tau)(\partial_U)[(\mathbf{R}\cdot\mathbf{U})*(m_o/\tau_{Planck})]$
$(m_o/\tau_{Planck})(\partial_R)[(\mathbf{R}\cdot\mathbf{U})] = (m_o/\tau_{Planck})(d/d\tau)(\partial_U)[(\mathbf{R}\cdot\mathbf{U})]$
$(\partial_R)[(\mathbf{R}\cdot\mathbf{U})] = (d/d\tau)(\partial_U)[(\mathbf{R}\cdot\mathbf{U})]$
$\mathbf{U} = (d/d\tau)\mathbf{R}$

$(d/d\tau)(\mathbf{P_T}) = (\partial_R)[H_o]$
$(d/d\tau)(\mathbf{P}+q\mathbf{A}) = (\partial_R)[\mathbf{P_T}\cdot\mathbf{U}]$
$\mathbf{F}+q(d/d\tau)\mathbf{A} = (\partial_R)[\mathbf{P}\cdot\mathbf{U}+q\mathbf{A}\cdot\mathbf{U}]$
$F^\alpha+q(d/d\tau)A^\alpha = (\partial^\alpha)[P^\beta U_\beta+qA^\beta U_\beta]$
$F^\alpha+q(U_\beta\partial^\beta)A^\alpha = [0^\alpha+q(\partial^\alpha)A^\beta U_\beta]$
$F^\alpha+qU_\beta\partial^\beta A^\alpha = qU_\beta\partial^\alpha A^\beta$
$F^\alpha = qU_\beta(\partial^\alpha A^\beta - \partial^\beta A^\alpha) = qU_\beta \boldsymbol{F}^{\alpha\beta}$: Lorentz EM Force Eqn. with $\boldsymbol{F}^{\alpha\beta}$ = Faraday EM Tensor

$H = \gamma(\mathbf{P_T} \cdot \mathbf{U})$ and $L = -(\mathbf{P_T} \cdot \mathbf{U})/\gamma$
$H_o = (\mathbf{P_T} \cdot \mathbf{U})$ and $L_o = -(\mathbf{P_T} \cdot \mathbf{U})$

$S = \int L dt = \int(\gamma/\gamma)L\, dt = \int(\gamma L)(dt/\gamma) = \int(L_o)(d\tau) = \int L_o\, d\tau = \int -(\mathbf{P_T} \cdot \mathbf{U})\, d\tau = -\int(\mathbf{P}+q\mathbf{A})\cdot\mathbf{U}\, d\tau = -\int(\mathbf{P}\cdot\mathbf{U}+q\mathbf{A}\cdot\mathbf{U})\, d\tau = -\int(E_o+q\mathbf{A}\cdot\mathbf{U})\, d\tau = -\int(E_o\, d\tau+q\mathbf{A}\cdot d\mathbf{R})$
$= -\int(E_o\, d\tau+q\mathbf{A}\cdot(d\mathbf{R}/d\tau)\, d\tau) = -\int(E_o\, d\tau+q\mathbf{A}\cdot\mathbf{U}\, d\tau) = -\int(E_o\, d\tau/dt+q\mathbf{A}\cdot\mathbf{U}\, d\tau/dt)dt = -\int(E_o/\gamma+q\mathbf{A}\cdot\mathbf{U}/\gamma)dt = -\int(m_o c^2/\gamma+q(\varphi - \mathbf{a}\cdot\mathbf{u}))dt$

Proof of Euler-Lagrange using the Least Action Principle: $\delta S = 0$ from Calculus of Variations
Essentially the integral along a worldline path is set so that the action S is minimized.

$\delta(L_o) = (\partial L_o/\partial\mathbf{R})\delta\mathbf{R} + (\partial L_o/\partial\mathbf{U})\delta\mathbf{U} = \partial_\mathbf{r}(L_o)\delta\mathbf{R} + \partial_\mathbf{u}(L_o)\delta\mathbf{U}$: Setting Lagrangian L_o in **R** & **U** coordinates

Product Differentiation Rule: $d(uv) = d(u)v + ud(v)$
Integrate all parts: $\int d(uv) = \int v\, du + \int u\, dv$: $uv| = \int v\, du + \int u\, dv$
Integration by parts: $\int u\, dv = uv| - \int v\, du$

$S = \int L_o\, d\tau$
$\delta S = 0$
$= \delta\int L_o\, d\tau$
$= \int\delta(L_o)d\tau$
$= \int[\partial_\mathbf{r}(L_o)\delta\mathbf{R} + \partial_\mathbf{u}(L_o)\delta\mathbf{U}]d\tau$
$= \int[\partial_\mathbf{r}(L_o)\delta\mathbf{R} + \partial_\mathbf{u}(L_o)\delta d\mathbf{R}/d\tau]d\tau$
$= \int[\partial_\mathbf{r}(L_o)\delta\mathbf{R} + \partial_\mathbf{u}(L_o)(d/d\tau)\delta\mathbf{R}]d\tau$
$= \int\partial_\mathbf{r}(L_o)\delta\mathbf{R}\, d\tau + \int\partial_\mathbf{u}(L_o)(d)(\delta\mathbf{R})$
Use Integration by parts on 2nd term... effectively moves the $(d/d\tau)$ to before the $\partial_\mathbf{u}(L_o)$
$= \int\partial_\mathbf{r}(L_o)\delta\mathbf{R}\, d\tau + \partial_\mathbf{u}(L_o)(\delta\mathbf{R})| - \int\delta\mathbf{R}\, d(\partial_\mathbf{u}(L_o))$
$= \int\partial_\mathbf{r}(L_o)\delta\mathbf{R}\, d\tau + \partial_\mathbf{u}(L_o)(\delta\mathbf{R})| - \int\delta\mathbf{R}\, d(\partial_\mathbf{u}(L_o))\, d\tau/d\tau$
$= \int\partial_\mathbf{r}(L_o)\delta\mathbf{R}\, d\tau + \partial_\mathbf{u}(L_o)(\delta\mathbf{R})| - \int\delta\mathbf{R}\, (d/d\tau)(\partial_\mathbf{u}(L_o))\, d\tau$
$= \int\partial_\mathbf{r}(L_o)\delta\mathbf{R}\, d\tau + [\ 0\ ] - \int\delta\mathbf{R}\, (d/d\tau)(\partial_\mathbf{u}(L_o))\, d\tau$
$= \int[\partial_\mathbf{r}(L_o) - (d/d\tau)(\partial_\mathbf{u}(L_o))]\, d\tau\, \delta\mathbf{R}$
$= 0$
which gives
$[\partial_\mathbf{r}(L_o) - (d/d\tau)(\partial_\mathbf{u}(L_o))] = 0$: So that it is true for any variation $\delta\mathbf{R}$ along the worldline $d\tau$
$[\partial_\mathbf{r} - (d/d\tau)(\partial_\mathbf{u})]L_o = 0$
$\partial_\mathbf{r} = (d/d\tau)(\partial_\mathbf{u})$
$(\partial_\mathbf{r})[L_o] = (d/d\tau)(\partial_\mathbf{u})[L_o]$ Relativistic Euler-Lagrange Eqn.

Conceptually easier Proof of Euler-Lagrange using 4-Vectors:
Instead of integration along a path, the local differential of Lorentz Scalar must take a particular form to maintain Lorentz Invariance.
$\mathbf{A} = (a^0,a^1,a^2,a^3)$: A generic 4-Vector
$\partial_\mathbf{a} = (\partial f/\partial a^0,-\partial f/\partial a^1,-\partial f/\partial a^2,-\partial f/\partial a^3)$: The 4-Gradient wrt. 4-Vector **A**
Minkowski Metric $\eta^{\mu\nu} = \mathrm{Diag}[+1,-1,-1,-1]$ in Cartesian, different in other coords

$df[a^0,a^1,a^2,a^3] = da^0(\partial f/\partial a^0) + da^1(\partial f/\partial a^1) + da^2(\partial f/\partial a^2) + da^3(\partial f/\partial a^3) = (d\mathbf{A}\cdot\partial_\mathbf{a})f$
$d[f]= d\mathbf{A}\cdot\partial_\mathbf{a}[f]$: The differential of a generic function f in **A** coordinates
$d[] = d\mathbf{A}\cdot\partial_\mathbf{a}[]$: In abstract operator form, which is Lorentz Invariant

$d=d$
$(d\mathbf{R}\cdot\partial_\mathbf{r}) = (d\mathbf{U}\cdot\partial_\mathbf{u})$ Essentially allowing the Lorentz Invariant in linearly-related coordinate systems **R** and **U**
$(d\mathbf{R}\cdot\partial_\mathbf{r}) = [d(d\mathbf{R}/d\tau)\cdot\partial_\mathbf{u}]$
$(d\mathbf{R}\cdot\partial_\mathbf{r}) = [(d)(d\mathbf{R})/d\tau\cdot\partial_\mathbf{u}]$
$(d\mathbf{R}\cdot\partial_\mathbf{r}) = [(d\mathbf{R})(d/d\tau)\cdot\partial_\mathbf{u}]$
$(d\mathbf{R}\cdot\partial_\mathbf{r}) = [d\mathbf{R}\cdot(d/d\tau)\partial_\mathbf{u}]$
$(\partial_\mathbf{r}) = (d/d\tau)\partial_\mathbf{u}$
$(\partial_\mathbf{r})[L_o] = (d/d\tau)(\partial_\mathbf{u})[L_o]$ Relativistic Euler-Lagrange Eqn.

Let $L_o = k(\mathbf{R}\cdot\mathbf{U})$: $(\partial_\mathbf{r})k(\mathbf{R}\cdot\mathbf{U}) = (d/d\tau)\partial_\mathbf{u}k(\mathbf{R}\cdot\mathbf{U})$: $k\mathbf{U} = (d/d\tau)k\mathbf{R}$: $\mathbf{U} = (d/d\tau)\mathbf{R}$ The Standard SR 4-Velocity,4-Position Relation
$(\partial_\mathbf{r})k(c^2\tau)=(d/d\tau)\partial_\mathbf{u}k(c^2\tau)$ comes from the Euler-Lagrange of ProperTime τ

Let $L_o = k(\mathbf{P}\cdot\mathbf{U})$: $(\partial_\mathbf{r})k(\mathbf{P}\cdot\mathbf{U}) = (d/d\tau)\partial_\mathbf{u}k(\mathbf{P}\cdot\mathbf{U})$: $k(\partial_\mathbf{r})E_o = k(d/d\tau)\mathbf{P}$: $\mathbf{0} = \mathbf{F}$ SR Force on a free particle is zero

Let $L_o = k(\mathbf{P}\cdot\mathbf{U})+q(\mathbf{A}\cdot\mathbf{U})$: $(\partial_\mathbf{r})[k(\mathbf{P}\cdot\mathbf{U})+q(\mathbf{A}\cdot\mathbf{U})] = (d/d\tau)\partial_\mathbf{u}[k(\mathbf{P}\cdot\mathbf{U})+q(\mathbf{A}\cdot\mathbf{U})]$: $q(\partial_\mathbf{r})(\mathbf{A}\cdot\mathbf{U}) = \mathbf{F} + q(\mathbf{U}\cdot\partial)\mathbf{A}$
$(d/d\tau)P^\alpha = F^\alpha = qU_\beta[\partial^\alpha A^\beta-\partial^\beta A^\alpha] = qU_\beta\boldsymbol{F}^{\alpha\beta}$: The 4D EM Lorentz Force Equation ($\boldsymbol{F}^{\alpha\beta}$ is the Faraday EM Tensor)
$(d/d\tau)\mathbf{p} = \gamma\mathbf{f} = q\gamma(\mathbf{e} + \mathbf{v}\times\mathbf{b})$: $(d/dt)\mathbf{p} = \mathbf{f} = q(\mathbf{e} + \mathbf{v}\times\mathbf{b})$: The 3D EM Lorentz Force Equation

Why there are only two time derivatives in most physical situations & Ostrogradsky Instability

$\mathbf{R}\cdot\mathbf{R} = (c\tau)^2$: $\mathbf{U}\cdot\mathbf{U} = c^2$: $\mathbf{A}\cdot\mathbf{A} = -\alpha^2$ $\quad d/d\tau(\mathbf{U}\cdot\mathbf{U}) = d/d\tau(c^2) = 0 = 2(\mathbf{U}\cdot\mathbf{A})$ $\quad d/d\tau(\mathbf{R}\cdot\mathbf{U}) = d/d\tau(c^2\tau) = c^2 = (\mathbf{U}\cdot\mathbf{U}) + (\mathbf{R}\cdot\mathbf{A})$

$d/d\tau(\mathbf{U}\cdot\mathbf{A}) = (\mathbf{A}\cdot\mathbf{A})+(\mathbf{U}\cdot\mathbf{J}) = d/d\tau(0) = 0$, so $\mathbf{U}\cdot\mathbf{J} = -\mathbf{A}\cdot\mathbf{A} = \alpha^2$, then $\mathbf{J} = (\alpha/c)^2\mathbf{U}$, meaning the 4-Jerk **J** is ~ to 4-Velocity **U**, using the same proper acceleration parameter α from the 4-Acceleration, so no new information. This gives: $\mathbf{J}\cdot\mathbf{A} = (\alpha^2/c^2)\mathbf{U}\cdot\mathbf{A} = (\alpha/c)^2(0) = 0$.

$d/d\tau(\mathbf{A}\cdot\mathbf{J}) = (\mathbf{J}\cdot\mathbf{J})+(\mathbf{A}\cdot\mathbf{S}) = d/d\tau(0) = 0$, so $\mathbf{A}\cdot\mathbf{S} = -\,\mathbf{J}\cdot\mathbf{J} = -(\alpha^2/c^2)^2(\mathbf{U}\cdot\mathbf{U}) = -\alpha^4/c^2$, then $\mathbf{S} = (\alpha/c)\mathbf{A}$, meaning the 4-Snap **S** is ~ to 4-Acceleration **A**, again using the same parameter α from the 4-Acceleration, so no new information. This gives: $\mathbf{S}\cdot\mathbf{J} = (\alpha/c)\mathbf{A}\cdot(\alpha/c)^2\mathbf{U} = (\alpha/c)^3\mathbf{A}\cdot\mathbf{U} = (\alpha/c)^3(0) = 0$.

Continue this downward ladder of derivatives. Each is 4D Orthogonal to the last, and proportional to the next-to-last.
So, physics is generally described by $\mathbf{R} = (ct,\mathbf{r})$: $\mathbf{U} = \gamma(c,\mathbf{u}) = d\mathbf{R}/d\tau$: $\mathbf{A} = \gamma(c\gamma',\gamma'\mathbf{u}+\gamma\mathbf{a}) = d\mathbf{U}/d\tau$: Hence, two time derivatives.

==

Let **B** be a physical 4-Vector
Denote all ProperTime Derivatives of **B**:
$\mathbf{B}^{n'} = (d/d\tau)^n\mathbf{B}$ $\qquad$ ex. $\mathbf{B}^{3'} = \mathbf{B}''' = (d/d\tau)^3\,\mathbf{B} = d^3\mathbf{B}/d\tau^3$
$(\mathbf{B}^{n'}\cdot\mathbf{B}^{n'})$ = Invariant magnitude-squared of $\mathbf{B}^{n'}$

Let $(\mathbf{B}^{n'}\cdot\mathbf{B}^{n+1'}) = 0$, meaning $(\mathbf{B}^{n'} \perp_{4D} \mathbf{B}^{n+1'})$, meaning $\mathbf{B}^{n'}$ is 4D orthogonal to $\mathbf{B}^{n+1'}$
Take the next ProperTime Derivative of this:
Then $(\mathbf{B}^{n'}\cdot\mathbf{B}^{n+1'})' = (\mathbf{B}^{n+1'}\cdot\mathbf{B}^{n+1'}) + (\mathbf{B}^{n'}\cdot\mathbf{B}^{n+2'}) = 0$
$(\mathbf{B}^{n'}\cdot\mathbf{B}^{n+2'}) = -\,(\mathbf{B}^{n+1'}\cdot\mathbf{B}^{n+1'})$
A solution is $\mathbf{B}^{n+2'} = g\mathbf{B}^{n'}$ $\qquad$ Another solution: $(b^0)^{n+1'} = (b^0)^{n+2'} = 0$ and $\mathbf{b}^{n+2'} = g\mathbf{b}^{n'}$
$(\mathbf{B}^{n'}\cdot g\mathbf{B}^{n'}) = g(\mathbf{B}^{n'}\cdot\mathbf{B}^{n'}) = -\,(\mathbf{B}^{n+1'}\cdot\mathbf{B}^{n+1'})$: $\quad g = -\,(\mathbf{B}^{n+1'}\cdot\mathbf{B}^{n+1'})/(\mathbf{B}^{n'}\cdot\mathbf{B}^{n'})$
Thus, $(\mathbf{B}^{n+2'}\cdot\mathbf{B}^{n+2'}) = g^2(\mathbf{B}^{n'}\cdot\mathbf{B}^{n'}) = (\mathbf{B}^{n+1'}\cdot\mathbf{B}^{n+1'})^2/(\mathbf{B}^{n'}\cdot\mathbf{B}^{n'})$
Importantly, $(\mathbf{B}^{n+2'}\cdot\mathbf{B}^{n+2'})/(\mathbf{B}^{n+1'}\cdot\mathbf{B}^{n+1'}) = (\mathbf{B}^{n+1'}\cdot\mathbf{B}^{n+1'})/(\mathbf{B}^{n'}\cdot\mathbf{B}^{n'})$, meaning the ratio remains the same between levels...
Since g is a function of $\mathbf{B}^{n+1'}$ and $\mathbf{B}^{n'}$, g is not a new independent component, it is a dependent component.
Since $\mathbf{B}^{n+2'} = g\mathbf{B}^{n'}$, then $(\mathbf{B}^{n'}\cdot\mathbf{B}^{n+1'}) = 0 = g(\mathbf{B}^{n'}\cdot\mathbf{B}^{n+1'}) = (g\mathbf{B}^{n'}\cdot\mathbf{B}^{n+1'}) = (\mathbf{B}^{n+2'}\cdot\mathbf{B}^{n+1'}) = (\mathbf{B}^{n+1'}\cdot\mathbf{B}^{n+2'}) = 0$

Since $(\mathbf{B}^{n+1'}\cdot\mathbf{B}^{n+2'}) = 0$, Repeat this process indefinitely…

For a physical system:
$\mathbf{B}^{0'} = \mathbf{R} = (d/d\tau)^0\mathbf{R}$ = 4-Position
$\mathbf{B}^{1'} = \mathbf{U} = (d/d\tau)^1\mathbf{R} = \mathbf{R}'$ = 4-Velocity
$\mathbf{B}^{2'} = \mathbf{A} = (d/d\tau)^2\mathbf{R} = \mathbf{R}''$ = 4-Acceleration
$\mathbf{B}^{3'} = \mathbf{J} = (d/d\tau)^3\mathbf{R} = \mathbf{R}'''$ = 4-Jerk
$\mathbf{B}^{4'} = \mathbf{S} = (d/d\tau)^4\mathbf{R} = \mathbf{R}''''$ = 4-Snap
...
The first occurrence of $(\mathbf{B}^{n'}\cdot\mathbf{B}^{n+1'}) = 0$ for a physical system is n=1
$\mathbf{B}^{1'} = \mathbf{U} = \mathbf{R}'$
$\mathbf{B}^{2'} = \mathbf{A} = \mathbf{R}''$
Thus, the first 4-Vector with a dependent component is $\mathbf{B}^{3'} = \mathbf{J}$
All further ProperTime Derivatives are likewise dependent components.
$\mathbf{U}\cdot\mathbf{U} = c^2$ $\qquad$ $\mathbf{A}\cdot\mathbf{A} = -\alpha^2$
The ratio is $(\mathbf{A}\cdot\mathbf{A})/(\mathbf{U}\cdot\mathbf{U}) = (-\alpha^2/c^2)$ for a hyperbolic system, which at the infinitesimal local level any system is...

4-Velocity	$\mathbf{U}\cdot\mathbf{U} = (c^2)$			
4-Acceleration	$\mathbf{A}\cdot\mathbf{A} = (-\alpha^2/c^2)\mathbf{U}\cdot\mathbf{U} = (-\alpha^2/c^2)(c^2) = (-\alpha^2)$	$\mathbf{U}\cdot\mathbf{A} = 0$		
4-Jerk	$\mathbf{J}\cdot\mathbf{J} = (-\alpha^2/c^2)\mathbf{A}\cdot\mathbf{A} = (-\alpha^2/c^2)(-\alpha^2) = (\alpha^4/c^2)$	$\mathbf{A}\cdot\mathbf{J} = 0$	$\mathbf{J} = -(-\alpha^2)/(c^2)\mathbf{U} = (\alpha^2/c^2)\mathbf{U}$	$\mathbf{J}\cdot\mathbf{J} = (\alpha^2/c^2)^2\mathbf{U}\cdot\mathbf{U} = (\alpha^4/c^2)$
4-Snap	$\mathbf{S}\cdot\mathbf{S} = (-\alpha^2/c^2)\mathbf{J}\cdot\mathbf{J} = (-\alpha^2/c^2)(\alpha^4/c^2) = (-\alpha^6/c^4)$	$\mathbf{J}\cdot\mathbf{S} = 0$	$\mathbf{S} = -(\alpha^4/c^2)/(-\alpha^2)\mathbf{A} = (\alpha^2/c^2)\mathbf{A}$	$\mathbf{S}\cdot\mathbf{S} = (\alpha^2/c^2)^2\mathbf{A}\cdot\mathbf{A} = (-\alpha^6/c^4)$
4-Crackle	$\mathbf{C}\cdot\mathbf{C} = (-\alpha^2/c^2)\mathbf{S}\cdot\mathbf{S} = (-\alpha^2/c^2)(-\alpha^6/c^4) = (\alpha^8/c^6)$	$\mathbf{S}\cdot\mathbf{C} = 0$	$\mathbf{C} = -(-\alpha^6/c^4)/(\alpha^4/c^2)\mathbf{J} = (\alpha^2/c^2)\mathbf{J}$	$\mathbf{C}\cdot\mathbf{C} = (\alpha^2/c^2)^2\mathbf{J}\cdot\mathbf{J} = (\alpha^8/c^6)$
4-Pop	$\mathbf{P}\cdot\mathbf{P} = (-\alpha^2/c^2)\mathbf{C}\cdot\mathbf{C} = (-\alpha^2/c^2)(\alpha^8/c^6) = (-\alpha^{10}/c^8)$	$\mathbf{C}\cdot\mathbf{P} = 0$	$\mathbf{P} = -(\alpha^8/c^6)/(-\alpha^6/c^4)\mathbf{S} = (\alpha^2/c^2)\mathbf{S}$	$\mathbf{P}\cdot\mathbf{P} = (\alpha^2/c^2)^2\mathbf{S}\cdot\mathbf{S} = (-\alpha^{10}/c^8)$

The ratio is (...) for a circular system

4-Velocity	$\mathbf{U}\cdot\mathbf{U} = (c^2)$			
4-Acceleration	$\mathbf{A}\cdot\mathbf{A} = -\gamma^4 a^2 = -(\alpha)^2$	$\mathbf{U}\cdot\mathbf{A} = 0$		
4-Jerk	$\mathbf{J}\cdot\mathbf{J} = -\gamma^6 j^2 = -(\iota)^2$	$\mathbf{A}\cdot\mathbf{J} = 0$		
4-Snap	$\mathbf{S}\cdot\mathbf{S} =$	$\mathbf{J}\cdot\mathbf{S} = 0$		
4-Crackle	$\mathbf{C}\cdot\mathbf{C} =$	$\mathbf{S}\cdot\mathbf{C} = 0$		
4-Pop	$\mathbf{P}\cdot\mathbf{P} =$	$\mathbf{C}\cdot\mathbf{P} = 0$		

Symmetric 4D-(2,0)-Tensor 'Fluid' Description of a Particle

Mass Density $\rho[x^\alpha] = \int(m_o/\sqrt{[-g[x^\alpha]]})\delta^4[x^\alpha - \gamma^\alpha[\tau]]d\tau$

$T^{\mu\nu}[x^\alpha] = \rho[x^\alpha](u^\mu)[\tau](u^\nu)[\tau]$ with $(u^\mu)=(d\gamma^\mu/d\tau)[\tau]$ with $\gamma^\mu[\tau]$ as the particle worldline

$T^{\mu\nu}[x^\alpha] = \rho[x^\alpha](d\gamma^\mu/d\tau)[\tau](d\gamma^\nu/d\tau)[\tau]$

$T^{\mu\nu}[x^\alpha] = \int(m_o/\sqrt{[-g[x^\alpha]]})(d\gamma^\mu/d\tau)[\tau](d\gamma^\nu/d\tau)[\tau]\delta^4[x^\alpha - \gamma^\alpha[\tau]]d\tau$

$T^{\mu\nu}[t,x^j] = \int(m_o/\sqrt{[-g[t,x^j]]})(d\gamma^\mu/d\tau)[t'](d\gamma^\nu/d\tau)[t']\delta(t-t')\delta^3[x^j - \gamma^j[t]](d\tau/dt)[t']dt'$

$T^{\mu\nu}[t,x^j] = (m_o/\sqrt{[-g[t,x^j]]})(d\gamma^\mu/d\tau)[t](d\gamma^\nu/d\tau)[t]\delta^3[x^j - \gamma^j[t]](d\tau/dt)[t]$

Mass conservation implies the alignment of the four-velocity with the worldline for a "single-particle" fluid

//=======================================

N-Differential $d\mathbf{R} = (cdt,dr^0,\ldots,dr^{N-1}) = (cdt,d\mathbf{r})$

$d\mathbf{R}\cdot d\mathbf{R} = (cdt,d\mathbf{r})\cdot(cdt,d\mathbf{r}) = c^2dt^2 - d\mathbf{r}\cdot d\mathbf{r} = c^2d\tau^2$

$d\mathbf{R}\cdot d\mathbf{R}/dt^2 = c^2dt^2/dt^2 - d\mathbf{r}\cdot d\mathbf{r}/dt^2 = c^2d\tau^2/dt^2$

$c^2 - \mathbf{u}\cdot\mathbf{u} = c^2d\tau^2/dt^2$

$1 - \mathbf{u}\cdot\mathbf{u}/c^2 = d\tau^2/dt^2$

$\sqrt{[1 - \mathbf{u}\cdot\mathbf{u}/c^2]} = d\tau/dt$

$1/\sqrt{[1 - \mathbf{u}\cdot\mathbf{u}/c^2]} = dt/d\tau = \gamma$

$d\mathbf{R}/d\tau = (d/d\tau)(cdt,d\mathbf{r}) = (dt/dt)(d/d\tau)(cdt,d\mathbf{r}) = (dt/d\tau)(d/dt)(cdt,d\mathbf{r}) = (dt/d\tau)(c,d\mathbf{r}/dt=\mathbf{u}) = (\gamma)(c,d\mathbf{r}/dt=\mathbf{u}) = \gamma(c,\mathbf{u}) = \mathbf{U}$

//=======================================

N-Differential $d\mathbf{R} = (cdt^0,\ldots,cdt^M , dr^0,\ldots,dr^{N-M}) = (cd\mathbf{t},d\mathbf{r})$ with possibility of multiple temporal dimensions

$d\mathbf{R}\cdot d\mathbf{R} = (cd\mathbf{t},d\mathbf{r})\cdot(cd\mathbf{t},d\mathbf{r}) = c^2d\mathbf{t}\cdot d\mathbf{t} - d\mathbf{r}\cdot d\mathbf{r} = c^2d\tau^2$

$d\mathbf{R}\cdot d\mathbf{R}/(d\mathbf{t}\cdot d\mathbf{t}) = c^2(d\mathbf{t}\cdot d\mathbf{t})/(d\mathbf{t}\cdot d\mathbf{t}) - d\mathbf{r}\cdot d\mathbf{r}/(d\mathbf{t}\cdot d\mathbf{t}) = c^2d\tau^2/(d\mathbf{t}\cdot d\mathbf{t})$

$c^2 - \mathbf{u}\cdot\mathbf{u} = c^2d\tau^2/(d\mathbf{t}\cdot d\mathbf{t})$ with $\mathbf{u}\cdot\mathbf{u} = (d\mathbf{r}\cdot d\mathbf{r})/(d\mathbf{t}\cdot d\mathbf{t})$

$1 - \mathbf{u}\cdot\mathbf{u}/c^2 = d\tau^2/(d\mathbf{t}\cdot d\mathbf{t})$

$\sqrt{[1 - \mathbf{u}\cdot\mathbf{u}/c^2]} = d\tau/\sqrt{(d\mathbf{t}\cdot d\mathbf{t})}$

$1/\sqrt{[1 - \mathbf{u}\cdot\mathbf{u}/c^2]} = \sqrt{(d\mathbf{t}\cdot d\mathbf{t})}/d\tau = \gamma$

$d\mathbf{R}/d\tau = (d/d\tau)(cd\mathbf{t},d\mathbf{r}) = (d\mathbf{t}/d\mathbf{t})(d/d\tau)(cd\mathbf{t},d\mathbf{r}) = (d\mathbf{t}/d\tau)(d/d\mathbf{t})(cd\mathbf{t},d\mathbf{r}) = (d\mathbf{t}/d\tau)(c,d\mathbf{r}/dt=\mathbf{u}) = (\gamma)(c,d\mathbf{r}/dt=\mathbf{u}) = \gamma(c,\mathbf{u}) = \mathbf{U}$

==

$\mathbf{R}\cdot\mathbf{U} = c^2\tau = \gamma(c^2t - \mathbf{r}\cdot\mathbf{u})$

$d/d\tau[\mathbf{R}\cdot\mathbf{U}] = d/d\tau[c^2\tau] = c^2$

$= d/d\tau[\mathbf{R}]\cdot\mathbf{U} + \mathbf{R}\cdot d/d\tau[\mathbf{U}]$

$= \mathbf{U}\cdot\mathbf{U} + \mathbf{R}\cdot\mathbf{A}$

$= c^2 + \mathbf{R}\cdot\mathbf{A}$

if $(\mathbf{R}\cdot\mathbf{U})$ is positive, then $\mathbf{R}$ and $\mathbf{U}$ are both time-like, then $(\mathbf{R}\cdot\mathbf{A}) = 0$

"Unit"Temporal 4-Vector $\overline{\mathbf{T}} = \gamma(1,\boldsymbol{\beta})$, with Lorentz Scalar Invariant $\overline{\mathbf{T}}\cdot\overline{\mathbf{T}} = T^\mu T_\mu = \gamma^2[1^2 - \boldsymbol{\beta}\cdot\boldsymbol{\beta}] = +1$ $\overline{\mathbf{T}} = \mathbf{U}/c$

Null 4-Vector $\overline{\mathbf{N}} \sim (\pm|\mathbf{a}|,\mathbf{a}) = a(\pm1,\hat{\mathbf{n}})$, with Lorentz Scalar Invariant $\overline{\mathbf{N}}\cdot\overline{\mathbf{N}} = N^\mu N_\mu = a^2[1^2 - \hat{\mathbf{n}}\cdot\hat{\mathbf{n}}] = 0$

"Unit"Spatial 4-Vector $\overline{\mathbf{S}} = \gamma_{\beta\hat{n}}(\beta\cdot\hat{n},\hat{\mathbf{n}})$, with Lorentz Scalar Invariant $\overline{\mathbf{S}}\cdot\overline{\mathbf{S}} = S^\mu S_\mu = \gamma_{\beta\hat{n}}{}^2[(\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^2 - \hat{\mathbf{n}}\cdot\hat{\mathbf{n}}] = -1$ $\overline{\mathbf{T}}\cdot\overline{\mathbf{S}} = (\gamma*\gamma_{\beta\hat{n}})[\boldsymbol{\beta}\cdot\hat{\mathbf{n}}-\boldsymbol{\beta}\cdot\hat{\mathbf{n}}] = 0$

$\overline{\mathbf{T}}\cdot\overline{\mathbf{S}} = 0 \leftrightarrow (\overline{T} \perp_{4D} \overline{S})$

Relativistic Motion: {4D General, 3D Circular=4D TimeHelix, 4D Hyperbolic, 4D Linear}

General SR Equations:

$\mathbf{R}^{n\prime} = d\mathbf{R}^{(n-1)\prime}/d\tau = d^{(n-1)}\mathbf{R}/d\tau^{(n-1)}$ $\quad \mathbf{J} = \mathbf{A}' = \mathbf{U}'' = \mathbf{R}'''$ $\quad {}' = d/d\tau = \gamma d/dt$ [SR Degrees of Freedom (DoF)]

$\mathbf{r}^{n\prime} = d\mathbf{r}^{(n-1)\prime}/dt = d^{(n-1)}\mathbf{r}/dt^{(n-1)}$ $\quad \mathbf{j} = \mathbf{a}' = \mathbf{u}'' = \mathbf{r}'''$ $\quad {}' = d/dt$ + [# of Constraints] = (10)

$(\mathbf{U}\cdot\mathbf{U}) = c^2$ is temporal, invariant, fundamental constant — due to Poincaré Group

$d/d\tau[\mathbf{U}\cdot\mathbf{U}] = d/d\tau[c^2] = 0 = d/d\tau[\mathbf{U}]\cdot\mathbf{U} + \mathbf{U}\cdot d/d\tau[\mathbf{U}] = 2(\mathbf{A}\cdot\mathbf{U}) = 0$

$(\mathbf{A}\cdot\mathbf{U} = 0) \leftrightarrow (\mathbf{A}\perp\mathbf{U})$: 4-Acceleration (normal to worldline) is orthogonal($\perp$) to 4-Velocity (tangent to worldline)

$d/d\tau[\mathbf{A}\cdot\mathbf{U}] = 0 = d/d\tau[\mathbf{A}]\cdot\mathbf{U} + \mathbf{A}\cdot d/d\tau[\mathbf{U}] = \mathbf{J}\cdot\mathbf{U} + \mathbf{A}\cdot\mathbf{A} = \mathbf{J}\cdot\mathbf{U} + -(\alpha)^2$

$(\mathbf{J}\cdot\mathbf{U}) = (\alpha)^2 = -(\mathbf{A}\cdot\mathbf{A})$

For a particle, one can always take $\mathbf{R} \to \mathbf{R} + \mathbf{R}_{init} = (ct,\mathbf{r}) + (ct_{init},\mathbf{r}_{init})$ with $\mathbf{R}_{init}$ = a constant 4-Vector due to Poincaré Invariance

General Motion: 10 independent variables = 10 DoF's

			All Lorentz Scalar Products are Invariants
4-Position	$\mathbf{R} = R^\mu = (ct,\mathbf{r})$	$= \mathbf{R}$	$(\mathbf{R}\cdot\mathbf{R}) = (ct)^2 - \mathbf{r}\cdot\mathbf{r} = (ct_o)^2 = (c\tau)^2 = -(\mathbf{r_o}\cdot\mathbf{r_o})$:either(±),variable
4-Velocity	$\mathbf{U} = U^\mu = \gamma(c,\mathbf{u})$	$= d\mathbf{R}/d\tau$	$(\mathbf{U}\cdot\mathbf{U}) = (c)^2$:temporal(+), fundamental constant
4-Acceleration	$\mathbf{A} = A^\mu = \gamma(c\gamma',\gamma'\mathbf{u}+\gamma\mathbf{a})$	$= d\mathbf{U}/d\tau$	$(\mathbf{A}\cdot\mathbf{A}) = -(a_o)^2 = -(\alpha)^2 = (i\alpha)^2$:spatial(-), variable
4-Jerk	$\mathbf{J} = J^\mu = \gamma(c(\gamma\gamma')',(\gamma\gamma'\mathbf{u}+\gamma^2\mathbf{a})')$	$= d\mathbf{A}/d\tau$	$(\mathbf{J}\cdot\mathbf{J}) = (c\gamma_o'')^2 - (j_o)^2$:either(±),variable
	$\mathbf{J} = J^\mu = \gamma(c(\gamma'^2+\gamma\gamma''),(\gamma'^2+\gamma\gamma'')\mathbf{u}+\gamma(3\gamma'\mathbf{a}+\gamma\mathbf{j}))$		

There are 10 DoF's: {3 **[acceleration]** & 3 **[velocity]** & 4 **[4-Position]**} matches 10 Poincaré symmetries:conservation laws
$(V^{\mu'} = \Lambda^{\mu'}{}_\nu V^\nu + \Delta V[\Delta X^{\mu'}])$ 10 = 6 Lorentz + 4 Translations

General Motion: (alt form **A**,**J**) using $\gamma' = \gamma^3\boldsymbol{\beta}'\cdot\boldsymbol{\beta} = \gamma^3(\mathbf{a}\cdot\mathbf{u})/c^2$

4-Acceleration $\mathbf{A} = A^\mu = (\gamma^4(\mathbf{a}\cdot\mathbf{u})/c,\gamma^4(\mathbf{a}\cdot\mathbf{u})\mathbf{u}/c^2+\gamma^2\mathbf{a})$ $\quad (\mathbf{A}\cdot\mathbf{A}) = -\gamma^6(\mathbf{a}\cdot\mathbf{u})^2/c^2 - \gamma^4\,\mathbf{a}\cdot\mathbf{a}$

4-Jerk $\mathbf{J} = J^\mu = \gamma(c(\gamma^6(\mathbf{a}\cdot\mathbf{u})^2/c^4 + \gamma^4[3\gamma^2(\mathbf{a}\cdot\mathbf{u})^2+(\mathbf{a}\cdot\mathbf{u})']/c^2),(\gamma^6(\mathbf{a}\cdot\mathbf{u})^2/c^4+\gamma^4[3\gamma^2(\mathbf{a}\cdot\mathbf{u})^2+(\mathbf{a}\cdot\mathbf{u})']/c^2\,)\mathbf{u}+\gamma(3\gamma^3(\mathbf{a}\cdot\mathbf{u})\mathbf{a}/c^2+\gamma\mathbf{j}))$

Imposed condition: Motion w/ Spatial Orthogonality $(\mathbf{a}\cdot\mathbf{u} = 0) \leftrightarrow (\mathbf{a}\perp\mathbf{u})$

4-Acceleration $\mathbf{A} = A^\mu = \gamma^2(0,\mathbf{a})_\perp$ if $(\mathbf{a}\cdot\mathbf{u})=0$ $\quad$ This also gives $\gamma' = \gamma^3(\mathbf{a}\cdot\mathbf{u})/c^2 \to 0$ which gives γ = constant

4-Jerk $\mathbf{J} = J^\mu = \gamma^3(0,\mathbf{j})_\perp$ if $(\mathbf{a}\cdot\mathbf{u})=0$

Circular Motion ×⦚ : constants{ |r| , |u| , |a| , |j| } ↔ { R , Ω , γ } w/ R=Radius, Ω=AngularFrequency, γ=Relativistic Gamma Factor
= Constant 3-vector-magnitudes Motion, but known as constant 3-acceleration-magnitude |a| = 3D Circular = 4D SR TimeHelix=LorentzHelix

4-Position $\mathbf{R} = R^\mu = (ct=c\gamma\tau,\mathbf{r} = R\hat{\mathbf{r}} = R(\cos[\Omega t+\theta_o]\hat{\mathbf{n}}_1+\sin[\Omega t+\theta_o]\hat{\mathbf{n}}_2)) = \mathbf{R} = d^0\mathbf{R}/d\tau^0$ $\quad \mathbf{R}\cdot\mathbf{R} = (ct^2) - \mathbf{r}\cdot\mathbf{r} = (ct^2)-R^2$

4-Velocity $\mathbf{U} = U^\mu = \gamma^1(c, \mathbf{u} = R\Omega\hat{\boldsymbol{\theta}} = R\Omega(-\sin[\Omega t+\theta_o]\hat{\mathbf{n}}_1+\cos[\Omega t+\theta_o]\hat{\mathbf{n}}_2)) = d\mathbf{R}/d\tau = d^1\mathbf{R}/d\tau^1$ $\quad \mathbf{U}\cdot\mathbf{U} = \gamma^2(c^2 - \mathbf{u}\cdot\mathbf{u}) = (c)^2$ $\quad 0<=R|\Omega|<=c$

4-Acceleration $\mathbf{A} = A^\mu = \gamma^2(0, \mathbf{a} = -R\Omega^2\hat{\mathbf{r}} = R\Omega^2(-\cos[\Omega t+\theta_o]\hat{\mathbf{n}}_1-\sin[\Omega t+\theta_o]\hat{\mathbf{n}}_2)) = d\mathbf{U}/d\tau = d^2\mathbf{R}/d\tau^2$ $\quad \mathbf{A}\cdot\mathbf{A} = \gamma^4(0^2 - \mathbf{a}\cdot\mathbf{a}) = -\gamma^4a^2 = -(\alpha)^2$

4-Jerk $\mathbf{J} = J^\mu = \gamma^3(0, \mathbf{j} = -R\Omega^3\hat{\boldsymbol{\theta}} = R\Omega^3(\sin[\Omega t+\theta_o]\hat{\mathbf{n}}_1-\cos[\Omega t+\theta_o]\hat{\mathbf{n}}_2)) = d\mathbf{A}/d\tau = d^3\mathbf{R}/d\tau^3$ $\quad \mathbf{J}\cdot\mathbf{J} = \gamma^6(0^2 - \mathbf{j}\cdot\mathbf{j}) = -\gamma^6j^2 = -(\iota)^2$

○ ↑ ⦚

Circular Motion follows path of ongoing Lorentz Transform **Λ→R** : (R)otation = Spatial Path+Const Temporal Motion = SR Helix

$\{\ |\mathbf{r}|=R\ ,\ |\mathbf{u}|=R\Omega\ ,\ |\mathbf{a}|=R\Omega^2\ ,\ |\mathbf{j}|=R\Omega^3\ \}$ are constants, $(\mathbf{a}\cdot\mathbf{u}) = 0$, $\gamma' = 0$, $\mathbf{a} = (-\Omega^2)\mathbf{r}$, $\mathbf{j} = (-\Omega^2)\mathbf{u}$, $d/d\tau = \gamma d/dt$, $\hat{\mathbf{n}}_1\cdot\hat{\mathbf{n}}_2 = 0$

There are 9 of 10 DoF's: ={1 [initial angle] & 1 [radius] & 3 **[acceleration]** & 4 **[4-Position=location in SpaceTime]**}
={ $1\ \theta_o$, 1 R , $1\ \Omega$, $2\ \hat{\mathbf{n}}_3 = \hat{\mathbf{n}}_1\times\hat{\mathbf{n}}_2$, 4 $(ct_{init},\mathbf{r}_{init})$}

The 1 constraint is: $\mathbf{a} = -(k^2)\mathbf{r} = \mathbf{r}'' = \ddot{\mathbf{r}}$, which gives { **r** , **u** , **a** , **j** } ~ (C^i * cos[kt] + S^i * sin[kt]) for each 3-vector

Boundary Conditions give k=Ω , $\mathbf{a} = -(\Omega^2)\mathbf{r}$, $\mathbf{j} = -(\Omega^2)\mathbf{u}$, $\mathbf{r} = R\hat{\mathbf{r}}$, $\mathbf{u} = R\Omega\hat{\boldsymbol{\theta}}$, $\mathbf{a} = -R\Omega^2\hat{\mathbf{r}}$, $(\mathbf{a}\cdot\mathbf{u}) = 0$, the (cos : sin) mathematics

$|\mathbf{a}| = |\mathbf{u}|^2/|\mathbf{r}| = (R\Omega^2) = (R\Omega)^2/(R)$ or $-\mathbf{a}\cdot\mathbf{r} = (R\Omega)^2 = \mathbf{u}\cdot\mathbf{u}$ $\qquad$ [9 DoF's] + [1 constraint] = (10)

Hyperbolic Motion ×(: constants { |R| , |U| , |A| , |J| } ↔ { D , c , α } w/ D=c²/α=Rindler "Distance", c=LightSpeed, α=ProperAccel
= Constant 4-Vector-Magnitudes Motion, but known as constant 4-Acceleration-magnitude |A| = 4D Hyperbolic

4-Position$_{\text{(0-base)}}$ $\mathbf{R} = R^\mu = (c^2/\alpha)(\sinh[\alpha\tau/c+\xi_o],(\cosh[\alpha\tau/c+\xi_o]-1)\ \hat{\mathbf{n}}) = \mathbf{R} = d^0\mathbf{R}/d\tau^0$ $\quad \mathbf{R}\cdot\mathbf{R} = (ct)^2 - [x+(c^2/\alpha)]^2 = -(c^2/\alpha)^2 = -D^2$

4-Position $\mathbf{R} = R^\mu = (c^2/\alpha)(\sinh[\alpha\tau/c+\xi_o],\cosh[\alpha\tau/c+\xi_o]\ \hat{\mathbf{n}}) = \mathbf{R} = d^0\mathbf{R}/d\tau^0$ $\quad \mathbf{R}\cdot\mathbf{R} = (c^2/\alpha)^2(\sinh^2 - \cosh^2\ \hat{\mathbf{n}}\cdot\hat{\mathbf{n}}) = -(c^2/\alpha)^2 = -D^2$

4-Velocity $\mathbf{U} = U^\mu = (c)(\cosh[\alpha\tau/c+\xi_o],\sinh[\alpha\tau/c+\xi_o]\ \hat{\mathbf{n}}) = d\mathbf{R}/d\tau = d^1\mathbf{R}/d\tau^1$ $\quad \mathbf{U}\cdot\mathbf{U} = (c)^2(\cosh^2 - \sinh^2\ \hat{\mathbf{n}}\cdot\hat{\mathbf{n}}) = (c)^2$

4-Acceleration $\mathbf{A} = A^\mu = (\alpha)(\sinh[\alpha\tau/c+\xi_o],\cosh[\alpha\tau/c+\xi_o]\ \hat{\mathbf{n}}) = d\mathbf{U}/d\tau = d^2\mathbf{R}/d\tau^2$ $\quad \mathbf{A}\cdot\mathbf{A} = (\alpha)^2(\sinh^2 - \cosh^2\ \hat{\mathbf{n}}\cdot\hat{\mathbf{n}}) = -(\alpha)^2$

4-Jerk $\mathbf{J} = J^\mu = (\alpha^2/c)(\cosh[\alpha\tau/c+\xi_o],\sinh[\alpha\tau/c+\xi_o]\ \hat{\mathbf{n}}) = d\mathbf{A}/d\tau = d^3\mathbf{R}/d\tau^3$ $\quad \mathbf{J}\cdot\mathbf{J} = (\alpha^2/c)^2(\cosh^2 - \sinh^2\ \hat{\mathbf{n}}\cdot\hat{\mathbf{n}}) = (\alpha^2/c)^2$

↑ ↔)×(

Hyperbolic Motion follows path of ongoing Lorentz Transform **Λ→B** : (B)oost = Time·Space Path = SR Hyperbolic

$\{\ |\mathbf{R}|=D=c^2/\alpha\ ,\ |\mathbf{U}|=c\ ,\ |\mathbf{A}|=\alpha\ ,\ |\mathbf{J}|=(\alpha^2/c)\ \}$ are constants, $(\mathbf{A}\cdot\mathbf{U})=0$, $\mathbf{A}=(\alpha^2/c^2)\mathbf{R}$, $\mathbf{J}=(\alpha^2/c^2)\mathbf{U}$, γ=cosh : $\gamma\beta$=sinh, $\tau'=d\tau/dt=1/\gamma$: $\tau=t/\gamma$

There are 8 of 10 DoF's: ={1 [initial hyperangle] & 3 **[proper acceleration]** & 4 **[4-Position=location in SpaceTime]**}
={ $1\ \xi_o$, $1\ \alpha$, $2\ \hat{\mathbf{n}}$, 4 $(ct_{init},\mathbf{r}_{init})$}

The constraint eqn. is: $\mathbf{A}=(k^2)\mathbf{R} = \mathbf{R}'' = d^2\mathbf{R}/d\tau^2$, which gives { **R** , **U** , **A** , **J** } ~ (C^μ * cosh[kτ] + S^μ * sinh[kτ]) for each 4-Vector

Boundary Conditions give k=(α/c) , $\mathbf{A}=(\alpha^2/c^2)\mathbf{R}$, $\mathbf{J}=(\alpha^2/c^2)\mathbf{U}$, $\mathbf{U}=(c)(\cosh[\alpha\tau/c],\sinh[\alpha\tau/c]\ \hat{\mathbf{n}})$, the (cosh : sinh) mathematics

The 2 constraints: $\mathbf{A}=(\alpha^2/c^2)\mathbf{R}$ splits into (temporal $a^0=(\alpha^2/c^2)r^0$: spatial $\mathbf{a}=(\alpha^2/c^2)\mathbf{r}$)

$|\mathbf{A}| = |\mathbf{U}|^2/|\mathbf{R}| = (\alpha) = (c)^2/(c^2/\alpha) = (c)^2/(D)$ or $-\mathbf{A}\cdot\mathbf{R} = (c)^2 = \mathbf{U}\cdot\mathbf{U}$ $\qquad$ [8 DoF's] + [2 constraints] = (10)

Relativistic Motion: {4D General, 3D Circular=4D TimeHelix, 4D Hyperbolic, 4D Linear}

SR 4D Linear Motion ×↗ : constants { **a** = **0** } with **a**=3-acceleration
= No Forces Minkowski Metric Motion = 4D Linear

4-Position	$\mathbf{R} = R^\mu = (ct + ct_{init}, \mathbf{r} = \mathbf{u}_{init}t + \mathbf{r}_{init})$	$= \mathbf{R} = d^0\mathbf{R}/d\tau^0$	$\mathbf{R}\cdot\mathbf{R} = (ct^2) - \mathbf{r}\cdot\mathbf{r} = (c\tau)^2$
4-Velocity	$\mathbf{U} = U^\mu = \gamma(c, \mathbf{u} = \mathbf{u}_{init})$	$= d\mathbf{R}/d\tau = d^1\mathbf{R}/d\tau^1$	$\mathbf{U}\cdot\mathbf{U} = \gamma^2(c^2 - \mathbf{u}\cdot\mathbf{u}) = (c)^2$
4-Acceleration	$\mathbf{A} = A^\mu = (0, \mathbf{a} = \mathbf{a}_{init} = \mathbf{0})$	$= d\mathbf{U}/d\tau = d^2\mathbf{R}/d\tau^2$	$\mathbf{A}\cdot\mathbf{A} = (0^2 - \mathbf{a}\cdot\mathbf{a}) = -(\alpha)^2 = 0$
4-Jerk	$\mathbf{J} = J^\mu = (0, \mathbf{j} = \mathbf{0})$	$= d\mathbf{A}/d\tau = d^3\mathbf{R}/d\tau^3$	$\mathbf{J}\cdot\mathbf{J} = (0^2 - \mathbf{j}\cdot\mathbf{j}) = (c\gamma_o'')^2 - (j_o)^2 = 0$

↑ ↔ ×↗

Linear Motion follows path of ongoing Lorentz Transform $\mathbf{\Lambda} \to \mathbf{I}_{[4]}$: (I)dentity = Time·Space Path = SR Linear
$(\mathbf{a}\cdot\mathbf{u}) = 0$, $\gamma' = 0$, $\gamma'' = 0$, γ = constant, $\mathbf{j} = \mathbf{0}$, $\mathbf{R} = \int \mathbf{U} d\tau \to \mathbf{U}\tau$ for no acceleration = inertial motion
There are 7 of 10 DoF's: ={3 **[initial velocity]** & 4 **[initial 4-Position=location in SpaceTime]**}
={ 3 $\mathbf{u}_{init}$, 4 $(ct_{init}, \mathbf{r}_{init})$}
The 3 constraints are: $\mathbf{a} = \mathbf{0}$, which splits into 3 separate components: $a^x = 0$, $a^y = 0$, $a^z = 0$
[7 DoF's] + [3 constraints] = (10)

To Reiterate, for 4D SR Motion [# DoF's] + [# Constraints] = 10 due to Poincaré Group = 10 due to Symmetric 4D (2,0)-Tensor:

General:		[10 DoF's] + [0 Constraints] = (10)	**A** = unconstrained
Circular=4DHelix:	×⦚	[9 DoF's] + [1 Constraint] = (10)	$\mathbf{A} = \gamma^2(0, \mathbf{a} = -R\Omega^2\hat{\mathbf{r}}) \leftrightarrow \mathbf{a} = -(\Omega^2)\mathbf{r}$
Hyperbolic:	×(	[8 DoF's] + [2 Constraints] = (10)	$\mathbf{A} = (\alpha^2/c^2)\mathbf{R}$ splits into (temporal $a^0 = (\alpha^2/c^2)r^0$: spatial $\mathbf{a} = (\alpha^2/c^2)\mathbf{r}$
Linear:	×↗	[7 DoF's] + [3 Constraints] = (10)	$\mathbf{A} = (0, \mathbf{0}) \leftrightarrow \mathbf{a} = \mathbf{0}$, which splits into $a^x = 0$, $a^y = 0$, $a^z = 0$

Physics Worksheet

$\mathbf{A} = (d/d\tau)\mathbf{U} = (d^2/d\tau^2)\mathbf{R} = (d/d\tau)(d/d\tau)\mathbf{R} = (\mathbf{U}\cdot\partial_\mathbf{r})(\mathbf{U}\cdot\partial_\mathbf{r})\mathbf{R}$

$(\partial_\mathbf{r}\cdot\partial_\mathbf{r})[\psi] = -(\mathbf{K}\cdot\mathbf{K})[\psi]$ for $\psi=e\hat{}(-i\mathbf{K}\cdot\mathbf{R})$
$(\mathbf{U}\cdot\partial_\mathbf{r})(\mathbf{U}\cdot\partial_\mathbf{r})[\mathbf{R}] = \mathbf{A}$
$(\mathbf{A}\cdot\partial_\mathbf{u})(\mathbf{U}\cdot\partial_\mathbf{r})[\mathbf{R}] = \mathbf{A}$
$(\mathbf{A}\cdot\partial_\mathbf{r})[\mathbf{R}] = \mathbf{A}$

$(\mathbf{U}\cdot\partial_\mathbf{r})[\mathbf{K}\cdot\mathbf{R}] = \mathbf{K}\cdot\mathbf{U} = (\omega/c,\mathbf{k})\cdot\gamma(c,\mathbf{u}) = \gamma(\omega - \mathbf{k}\cdot\mathbf{u}) = \omega_o$
$(\mathbf{U}\cdot\partial_\mathbf{r}) = (d\mathbf{R}/d\tau\cdot\partial_\mathbf{r}) = d/d\tau$
$(d\mathbf{R}\cdot\partial_\mathbf{r})[] = d[]$

$(d\mathbf{K}\cdot\partial_\mathbf{k})[] = d[]$

$(\mathbf{U}\cdot\partial_\mathbf{r})\mathbf{R} = \mathbf{U}$
$(\mathbf{K}\cdot\partial_\mathbf{r})\mathbf{R} = \mathbf{K}$

$\mathbf{K} = (\omega_o/c^2)\mathbf{U} = (\omega/c,\mathbf{k}) = (\omega_o/c^2)\gamma(c,\mathbf{u}) = \gamma(\omega_o/c^2)(c,\mathbf{u}) = (\omega/c^2)(c,\mathbf{u}) = (\omega/c,\mathbf{k}=\omega\mathbf{u}/c^2)$
$\mathbf{K}\cdot\mathbf{K} = (\omega_o/c^2)^2\mathbf{U}\cdot\mathbf{U} = (\omega_o/c)^2$
$\partial\cdot\partial = \partial_\mathbf{r}\cdot\partial_\mathbf{r} = b$ = Constant Invariant
$\partial_\mathbf{k}[\mathbf{K}\cdot\mathbf{R}] = \mathbf{R}$
$\partial_\mathbf{r}[\mathbf{K}\cdot\mathbf{R}] = \mathbf{K}$
$\partial_\mathbf{r}[e\hat{}(-i\mathbf{K}\cdot\mathbf{R})] = -i\mathbf{K}e\hat{}(-i\mathbf{K}\cdot\mathbf{R})$
$\partial_\mathbf{r}\cdot\partial_\mathbf{r}[e\hat{}(-i\mathbf{K}\cdot\mathbf{R})] = (-i\mathbf{K})^2e\hat{}(-i\mathbf{K}\cdot\mathbf{R})$
$(\partial_\mathbf{r}\cdot\partial_\mathbf{r})[e\hat{}(-i\mathbf{K}\cdot\mathbf{R})] = -(\mathbf{K}\cdot\mathbf{K})e\hat{}(-i\mathbf{K}\cdot\mathbf{R}) =$
$(\partial_\mathbf{r}\cdot\partial_\mathbf{r})[\psi] = -(\mathbf{K}\cdot\mathbf{K})[\psi]$ for $\psi=e\hat{}(-i\mathbf{K}\cdot\mathbf{R})=e\hat{}-i(\omega t-\mathbf{k}\cdot\mathbf{r}) = \psi[\mathbf{K},\mathbf{R}]$
$(\partial_\mathbf{k}\cdot\partial_\mathbf{k})[\psi] = -(\mathbf{R}\cdot\mathbf{R})[\psi]$ for $\psi=e\hat{}(-i\mathbf{K}\cdot\mathbf{R})$
$(\partial_\mathbf{r}\cdot\partial_\mathbf{k})[\psi] = -(\mathbf{R}\cdot\mathbf{K})[\psi]$ for $\psi=e\hat{}(-i\mathbf{K}\cdot\mathbf{R})$
$(\mathbf{U}\cdot\partial_\mathbf{r})[\psi] = -i(\mathbf{U}\cdot\mathbf{K})[\psi] = -i\omega_o[\psi]$
$(\mathbf{U}\cdot\partial_\mathbf{r})(\mathbf{U}\cdot\partial_\mathbf{r})[\psi] = (-i)^2(\mathbf{U}\cdot\mathbf{K})^2[\psi] = -(\omega_o)^2[\psi]$

$(\partial_\mathbf{r}\cdot\partial_\mathbf{r})$ $[1/m^2]$
$(\mathbf{U}\cdot\partial_\mathbf{r})(\mathbf{U}\cdot\partial_\mathbf{r})$ $[1/s^2]$

$(\mathbf{U}\cdot\partial_\mathbf{r})(\mathbf{U}\cdot\partial_\mathbf{r})\mathbf{R} = \mathbf{A}$ $[m/s^2]$
$(\mathbf{U}\cdot\partial_\mathbf{r})(\mathbf{U}\cdot\partial_\mathbf{r})[\ \psi[\mathbf{R}]\ ] = (-i)^2(\mathbf{U}\cdot\mathbf{K})^2[\psi] = -(\omega_o)^2[\psi]$ $[1/s^2]$

$(\mathbf{U}\cdot\mathbf{U})(\partial_\mathbf{r}\cdot\partial_\mathbf{r})[\psi] = (c)^2(-i)^2(\mathbf{K}\cdot\mathbf{K})^2[\psi] = -(\omega_o)^2[\psi]$ $[1/s^2]$
$(\mathbf{U}\cdot\partial_\mathbf{r})(\mathbf{U}\cdot\partial_\mathbf{r})[\psi] = (-i)^2(\mathbf{U}\cdot\mathbf{K})^2[\psi] = -(\omega_o)^2[\psi]$ $[1/s^2]$
$(\partial_\mathbf{r}\cdot\partial_\mathbf{r})[\psi] = (-i)^2(\mathbf{K}\cdot\mathbf{K})[\psi] = -(\omega_o/c)^2[\psi]$ $[1/m^2]$

$\mathbf{R} = (\mathbf{R}\cdot\partial_\mathbf{u})\mathbf{U} = (\mathbf{R}\cdot\partial_\mathbf{u})(\mathbf{U}\cdot\partial_\mathbf{a})\mathbf{A}$

$\mathbf{P_T} = -\partial_\mathbf{r}[S] = (H/c,\mathbf{p_T}) = -(\partial_t/c,-\nabla)[S] = (-\partial_t/c,\nabla)[S]$
$\mathbf{R} = (\mathbf{R}\cdot\partial_\mathbf{PT})\mathbf{P_T}$

Given all this, you might still wonder why the Laplacian is so common. It's simply because there are so few ways to write down partial differential equations that are low-order in time derivatives (required by Newton's second law, or at a deeper level, because Lagrangian mechanics is otherwise pathological), low-order in spatial derivatives, linear, translationally invariant, time invariant, and rotationally symmetric. There are essentially only five possibilities: the heat/diffusion, wave, Laplace, Schrodinger, and Klein-Gordon equations, and all of them involve the Laplacian.

I think a rule of thumb would be to start looking for the simplest Lagrangian you can think of. In the general case, a good Lagragian should obey homogeneity of space, time and isotropy of space which means that it can't contain explicitly the position, time and velocity, respectively. Then, the simplest allowed possibility is to have a Lagrangian with a velocity squared. Since we don't need to look for more conditions to be fulfilled, there is no need to add terms involving higher derivatives or combinations of other terms.

D'Alembertian: It is the generalization of the Laplace operator in the sense that it is the differential operator which is invariant under the isometry group of the underlying space and it reduces to the Laplace operator if restricted to time-independent functions. The overall sign of the metric here is chosen such that the spatial parts of the operator admit a negative sign, which is the usual convention in high-energy particle physics. The D'Alembert operator is also known as the wave operator because it is the differential operator appearing in the wave equations, and it is also part of the Klein–Gordon equation, which reduces to the wave equation in the massless case.

The additional factor of c in the metric is needed in physics if space and time are measured in different units; a similar factor would be required if, for example, the x direction were measured in meters while the y direction were measured in centimeters. Indeed, theoretical physicists usually work in units such that c = 1 in order to simplify the equation.

The d'Alembert operator generalizes to a hyperbolic operator on pseudo-Riemannian manifolds.
The isometry group of Minkowski space is the Poincaré group.

Even more interesting is when you then apply my good ole 4-vectors.
4-Position = $\mathbf{X} = d^0\mathbf{X}/d\tau^0$
4-Velocity = $\mathbf{U} = d^1\mathbf{X}/d\tau^1$
4-Acceleration $\mathbf{A} = d^2\mathbf{X}/d\tau^2$

Now time and space dimensions are explicitly captured by the 4-Vectors. Time becomes just another component of the 4-Position. Each 4-Vector has, on first glance, 4 independent components. So, 3 SR Vectors*4 components each =12. So, it would appear that we have 2 extra parameters not apparent in the classical vectors + time derivation.

However, SR to the rescue!

There are actually two constraint equations built-in to SR to bring the total back down to 10 independent parameters.
$\mathbf{U}\cdot\mathbf{U} = c^2$
$\mathbf{U}\cdot\mathbf{A} = 0$ $\mathbf{U}\perp\mathbf{A}$

I think that is a beautiful result! Not only is the time (t) incorporated into the 4-Position, but we also get a fundamental constant LightSpeed (c) out of the deal, and that 4-Velocity **U** is temporal, 4-Acceleration **A** is spatial, describing worldlines.

4-Velocity **U** is tangential to the worldline at every point and 4-Acceleration **A** is normal to the worldline at every point.
That the 4-Vector version, remembering that they are tensorial, also works with the symmetric 2-index tensor, GR's stress-energy, places the arguments on a much more rigorous level.

Now, are there situations in which the higher order derivatives become important? Perhaps... But see the **Ostrogradsky instability**.

They only become important for cases in which one or more of the independent parameters is transferred from our usual 3-vectors in such a way that you still only get 10 total independent parameters.

Even in the case of the **Norton Dome**. The non-zero snap parameter is a property of the curve shape. This translates into a particular time-varying acceleration parameter in the outwards-from-axis **x**-direction. The downward **z**-acceleration is still just gravity, and the **y**-direction is just acceleration = zero. So, you still have only at most 10 independent parameters describing the motion.

(3) N-Vectors used to make a Symmetric (2,0)-Tensor describing physical system (N-Position **R**, N-Velocity **U**, N-Acceleration **A**)
3N - # of constraints C = N(N+1)/2 independent components of symmetric 2-index tensor.
$3N - C = N(N+1)/2 = N^2/2 + N/2$
$6N - 2C = N^2 + N$
$N^2 - 5N + 2C = 0$
N = (5±√[25-4*2C])/2 = (5±√[25-8C])/2

if C=0, N = (5±√[25])/2 = (5±5)/2 = {0,5}
if C=1, N = (5±√[17])/2 = non-integer
if C=2, N = (5±√[9])/2 = (5±3)/2 = {1,4} **** $\mathbf{U}\cdot\mathbf{U} = c^2$ & $\mathbf{U}\cdot\mathbf{A} = 0$ **** SpaceTime is 4D
if C=3, N = (5±√[1])/2 = (5±1)/2 = {2,3}
if C>=4, N = complex

(2) N-Vectors used to make an AntiSymmetric (2,0)-Tensor describing physical system (**M** = **X^P** , **F** = **∂^A**)
2N - # of constraints C = N(N–1)/2 independent components of antisymmetric 2-index tensor.
$2N - C = N(N-1)/2 = N^2/2 - N/2$
$4N - 2C = N^2 - N$
$N^2 - 5N + 2C = 0$
N = (5±√[25-4*2C])/2 = (5±√[25-8C])/2

if C=0, N = (5±√[25])/2 = (5±5)/2 = {0,5}
if C=1, N = (5±√[17])/2 = non-integer
if C=2, N = (5±√[9])/2 = (5±3)/2 = {1,4} **** $F^{\mu\nu}F_{\mu\nu}=2\{(\mathbf{b}\cdot\mathbf{b})-(\mathbf{e}\cdot\mathbf{e}/c^2)\}$ & $\mathrm{Det}[F^{\mu\nu}]=\{(\mathbf{e}\cdot\mathbf{b})/c\}^2$ **** SpaceTime is 4D
if C=3, N = (5±√[1])/2 = (5±1)/2 = {2,3}
if C>=4, N = complex

{1} Anti-Symmetric 4D (2,0)-Tensor + {1} 4-Vector = 4D (1,0)-Tensor
===

[→] 4-LinearMomentum 4-Vector : 4 Independent parameters
$P^\mu = (E/c = mc, \mathbf{p} = m\mathbf{u}) = \mathbf{P}$ [kg·m/s = N·s]

P^μ has one invariant:
$(\mathbf{P}\cdot\mathbf{P}) = P^\alpha P_\alpha = (E/c)^2 - \mathbf{p}\cdot\mathbf{p} = (E_o/c)^2$

This invariant depends only on the scalar mass:energy E_o of the particle, not the velocity,
i.e. linear translation properties are relative, not absolute.
In other words, the 3-velocity or 3-momentum can always be boosted to spatial zero.

===

[↻] 4-AngularMomentum 4-Tensor,$_{\text{Antisymmetric}}$: $M^{\alpha\beta} = -M^{\beta\alpha}$: 6 Independent parameters
$M^{\alpha\beta} = X^\alpha P^\beta - X^\beta P^\alpha = \mathbf{X}\wedge\mathbf{P}$ [kg·m^2/s = N·m·s = J·s = Action]
→
$[M^{tt}\ M^{tx}\ M^{ty}\ M^{tz}]$
$[M^{xt}\ M^{xx}\ M^{xy}\ M^{xz}]$
$[M^{yt}\ M^{yx}\ M^{yy}\ M^{yz}]$
$[M^{zt}\ M^{zx}\ M^{zy}\ M^{zz}]$
=
$[0\ {-cn^x}\ {-cn^y}\ {-cn^z}]$
$[+cn^x\ 0\ {+l^z}\ {-l^y}]$
$[+cn^y\ {-l^z}\ 0\ {+l^x}]$
$[+cn^z\ {+l^y}\ {-l^x}\ 0]$
=
$[\ 0\ \ ,\ -cn^{0j}\ ]$
$[\ +cn^{i0},\ \varepsilon^{ij}{}_k l^k\ ]$
=
$[\ 0\ \ ,\ -c\mathbf{n}\ ]$
$[+c\mathbf{n}^T,\ \mathbf{x}\wedge\mathbf{p}\ ]$

$M^{\mu\nu}$ has two invariants:
$M_{\mu\nu}M^{\mu\nu} = \Sigma_{\mu=\nu}[M^{\mu\nu}]^2 - 2\Sigma_i[M^{i0}]^2 + 2\Sigma_{i>j}[M^{ij}]^2 = (0) - 2(c^2\mathbf{n}\cdot\mathbf{n}) + 2(\mathbf{l}\cdot\mathbf{l}) = 2\{(\mathbf{l}\cdot\mathbf{l})-(c^2\mathbf{n}\cdot\mathbf{n})\}$
$\text{Det}[M^{\mu\nu}] = \text{Pfaffian}[M^{\mu\nu}]^2 = [(-cn^x)(+l^x) - (-cn^y)(-l^y) + (-cn^z)(+l^z)]^2 = [-(cn^x l^x) - (cn^y l^y) - (cn^z l^z)]^2 = \{c(\mathbf{n}\cdot\mathbf{l})\}^2$

These invariants depend on the 3-angular-momentum **l**, and hence the rotational velocity **ω**,
via 3-angular_momentum relations
$\mathbf{l} = \boldsymbol{I}\cdot\boldsymbol{\omega}$ or $l^j = I^j{}_k\,\omega^k$
with ***I*** as the Inertia Tensor.

Since this relation is a function of angular velocity, and 3-angular_velocity **ω** cannot in general be boosted to spatial zero like the linear velocity,
4D Rotation is thus an absolute property.

Something is either rotating with a non-zero 4-AngularMomentum (At least one component of $M^{\mu\nu} \neq 0$, with boosts changing other components)
or it is not rotating (when $\mathbf{l} = \mathbf{n} = \mathbf{0}$ and hence $M^{\mu\nu} = 0^{\mu\nu}$).