Bell's Spaceship Paradox: A Tensor Analysis https://en.wikipedia.org/wiki/Bell's_spaceship_paradox

Consider two identically-constructed ships (same lengths, same engine thrusts, same accelerations) at rest in an inertial start frame. They face the same direction, one behind the other, along the same axis. The ships are connected to one another by a taut string, aft to bow. At a prearranged time in the start frame, both ships simultaneously thrust, and maintain identical accelerations in that frame. Their velocities with respect to the initial frame are always equal throughout the remainder of the experiment. This means, by definition, *that with respect to the initial frame,* the distance between the two ships does not change, even when they speed up to relativistic velocities. However, this setup leads to the ships having a seemingly relativistically-altered distance from one another in their own frames, due to Lorentz Contraction. In the start frame, the string appears to stay taut and connected; in the boosted frame, does the string snap? **What is the reality? For the paradox setup, the tensor math says it does NOT break :)**

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This can be examined using SR tensors, which are singularly adept at handling spacetime frames-of-reference and invariances.
The physical 4-Vectors used in SR are 4D (1,0)-Tensors, meaning (1 $^{\text{upper}}$, 0 $_{\text{lower}}$) indexes, and are geometrically Lorentz invariant.
4-Position $\mathbf{R} = R^\mu = (ct,\mathbf{r}) = (ct,x,y,z)$. It has internal components of time (t), & 3-position $\mathbf{r} = r^k = (x,y,z)$, which may vary with frame.
4-Displacement $\Delta\mathbf{R} = \Delta R^\mu = (c\Delta t,\Delta\mathbf{r}) = (ct_2\text{-}ct_1,\mathbf{r_2}\text{-}\mathbf{r_1}) = \mathbf{R_2}\text{-}\mathbf{R_1}$, and is fully Poincaré Invariant (=Lorentz + SpaceTime Translations).

Poincaré Invariance gives affine (=linear+translation) transformations between inertial frames for generic 4-Vectors **A** & **B**:
$A^{\mu'} = \Lambda^\mu{}_\nu A^\nu + S^\mu$ $\quad \mathbf{A} = A^\mu = (a^0,a^1,a^2,a^3) = (a^t,a^x,a^y,a^z)_{\text{Cartesian/Rectangular}} = (a^t,a^r,a^\theta,a^z)_{\text{Cylindrical}}$ Components vary in different reference-frames
$B^{\mu'} = \Lambda^\mu{}_\nu B^\nu + S^\mu$ $\quad \mathbf{B} = B^\mu = (b^0,b^1,b^2,b^3) = (b^t,b^x,b^y,b^z)_{\text{Cartesian/Rectangular}} = (b^t,b^r,b^\theta,b^z)_{\text{Cylindrical}}$ but each type references the same 4-Vector
$\mathbf{\Lambda} = \Lambda^\mu{}_\nu$ is a Lorentz Transform (a matrix or dyadic) with $\text{Det}[\Lambda^\mu{}_\nu] = \pm1$, and $\mathbf{S} = S^\mu$ is a SpaceTime Translation (also a 4-Vector).
Poincaré Transformations give one equation for each dimension $\{t,x,y,z\}$referenced by a Greek tensor index:

Each of these equations is just like a standard affine (=linear+translation) transform along a single line or dimension:
$x' = C\,x + D$ with C the linear, multiplicative component and D the translational, additive component.

4-Displacement $\Delta R^{\mu'}=(\Lambda^\mu{}_\nu B^\nu + S^\mu)\text{-}(\Lambda^\mu{}_\nu A^\nu + S^\mu)=(\Lambda^\mu{}_\nu B^\nu)\text{-}(\Lambda^\mu{}_\nu A^\nu)=\Lambda^\mu{}_\nu (B^\nu\text{-}A^\nu)=\Lambda^\mu{}_\nu (\Delta R^\nu)$ is independent of S^μ = translation invariant.

There is a 4D Invariant Inner Product for 4-Vectors, which is just one of many Tensor Invariants. See also the Trace & Determinant.
Using tensor gymnastics and the properties of the dimensionless Minkowski Metric $\boldsymbol{\eta}=\eta_{\alpha\beta}$ & dimensionless Lorentz Transform $\mathbf{\Lambda}=\Lambda^\mu{}_\nu$:
$\mathbf{A'}\cdot\mathbf{B'} = A^{\mu'}\eta_{\mu'\nu'}B^{\nu'} = +(a^{0'}b^{0'})\text{-}(a^{1'}b^{1'})\text{-}(a^{2'}b^{2'})\text{-}(a^{3'}b^{3'})$
$(\Lambda^{\mu'}{}_\alpha A^\alpha)\,\eta_{\mu'\nu'}(\Lambda^{\nu'}{}_\beta B^\beta) = (\Lambda^{\mu'}{}_\alpha\eta_{\mu'\nu'}\Lambda^{\nu'}{}_\beta)\,A^\alpha B^\beta = (\Lambda_{\nu'\alpha}\Lambda^{\nu'}{}_\beta)\,A^\alpha B^\beta = (\eta_{\rho\alpha}\Lambda^\rho{}_{\nu'}\Lambda^{\nu'}{}_\beta)\,A^\alpha B^\beta = (\eta_{\alpha\rho}\delta^\rho{}_\beta)\,A^\alpha B^\beta = (\eta_{\alpha\beta})\,A^\alpha B^\beta = A^\alpha(\eta_{\alpha\beta})B^\beta =$
$A^\alpha\eta_{\alpha\beta}B^\beta = \mathbf{A}\cdot\mathbf{B} = +(a^0b^0)\text{-}(a^1b^1)\text{-}(a^2b^2)\text{-}(a^3b^3)$ and hence $\mathbf{A}\cdot\mathbf{A} = +(a^0)^2\text{-}(a^1)^2\text{-}(a^2)^2\text{-}(a^3)^2$

For the 4-Displacement $\Delta\mathbf{R}$, this gives the familiar SpaceTime Invariant Interval between any two events at 4-Positions $\mathbf{R_1}$ and $\mathbf{R_2}$:
$\Delta\mathbf{R'}\cdot\Delta\mathbf{R'}=\Delta R^{\mu'}\Delta R_{\mu'}=\Delta R^\nu\Delta R_\nu=\Delta R^\alpha\eta_{\alpha\beta}\Delta R^\beta=\Delta\mathbf{R}\cdot\Delta\mathbf{R}=+(c\Delta t')^2\text{-}(\Delta x')^2\text{-}(\Delta y')^2\text{-}(\Delta z')^2=+(c\Delta t)^2\text{-}(\Delta x)^2\text{-}(\Delta y)^2\text{-}(\Delta z)^2$ = Same for all frames.

One beauty of the tensor formalism is that the Lorentz Transform Matrix $\mathbf{\Lambda} = \Lambda^\mu{}_\nu$ can apply to space-space transforms (**Λ→R**otations) or to space-time transforms (**Λ→B**oosts), or combinations. For example, the $\Lambda^\mu{}_\nu$ can be set to trade spatial-x with spatial-y (rotations), or to trade spatial-x with temporal-t (boosts), or to several more complex cases including combinations of rotations, boosts, and discrete reversals. In addition to the Lorentz Transform, Poincaré gives a SpaceTime Translation. This is key to solving the paradox.

Note that the full $\mathbf{\Lambda} = \Lambda^\mu{}_\nu$ Transform is over all 4 dimensions. We are looking at subsets of just two components in these examples.

For (x,y)-Rotation, one uses **Λ→R**otation(θ):
[x'] = [cos(θ) -sin(θ)][x]
[y'] [sin(θ) cos(θ)][y]

For (t,x)-Boost, one uses **Λ→B**oost(φ or β): with $\gamma=1/\sqrt{[1\text{-}\beta^2]}$ **:** $\cosh(\varphi)=\gamma$ **:** $\sinh(\varphi)=\gamma\beta$ **:** $\tanh(\varphi)=\beta$
[ct'] = [cosh(φ) -sinh(φ)][ct] = [γ -γβ][ct]
[x'] [-sinh(φ) cosh(φ)][x] = [-γβ γ][x]

For (x,y)-Identity, one uses **Λ→I**dentity:
[x'] = [1 0][x]
[y'] [0 1][y]

For a (t,x)-Spacetime Translation, one uses the first two components of $\mathbf{S} = (s^t,s^x,s^y,s^z)$:
[ct'] = [ct] + [s^t]
[x'] [x] [s^x]

The following are invariants, those scalars which observers in all frames must agree on, even though internal components may differ:
Full Lorentz $\mathbf{\Lambda}$: $+(ct')^2 - (x')^2 - (y')^2 - (z')^2 = +(ct)^2 - (x)^2 - (y)^2 - (z)^2$ because Determinant$[\Lambda^\mu{}_\nu] = +1$ for continuous, proper
Rotations $\mathbf{\Lambda}\to\mathbf{R}(x,y)$: $- (x')^2 - (y')^2 = - (x)^2 - (y)^2$ because Determinant$[R^\mu{}_\nu] = 1*(\cos^2 + \sin^2)*1 = +1$
Boosts $\mathbf{\Lambda}\to\mathbf{B}(t,x)$: $+(ct')^2 - (x')^2 = +(ct)^2 - (x)^2$ because Determinant$[B^\mu{}_\nu] = (\cosh^2 - \sinh^2)*1*1 = +1$

$\mathbf{R}\cdot\mathbf{R} = R^\alpha\eta_{\alpha\beta}R^\beta = +(ct)^2 - (x)^2 - (y)^2 - (z)^2$
The difference in signs comes from: 4D "Flat" <Time·Space> SR:Minkowski Metric $\eta_{\mu\nu} = \eta^{\mu\nu} \to$Diagonal$[+1,-1,-1,-1]$(Cartesian coordinates)
One can also reverse all signs, ie. use the alternate reverse signature, which is commonly seen for $(x')^2 + (y')^2 = (x)^2 + (y)^2$

Interestingly, the Tensor Invariant Trace$[\Lambda^\mu{}_\nu]$ identifies the transform type. This actually leads into CPT Symmetry.
Trace$[\Lambda^\mu{}_\nu]$: $2(1+\cos(\theta))=\{0..4\}$ are **R**otations, $(1+1+1+1)=\{4\}$ is **I**dentity, $2(1+\cosh(\varphi))=\{4..\text{Infinity}\}$ are **B**oosts.
This also makes sense because a **R**otation(θ=0) = **I**dentity I_4 = **B**oost(φ =0). The **R**otation meets the **B**oost at the **I**dentity.

General Motion using 4-Vectors: All Lorentz Scalar Products are Invariants

4-Position	$\mathbf{R} = R^\mu = (ct,\mathbf{r})$	$= \mathbf{R} = d^0\mathbf{R}/d\tau^0$	$(\mathbf{R}\cdot\mathbf{R}) = (ct)^2 - \mathbf{r}\cdot\mathbf{r} = (ct_o)^2 = (c\tau)^2 = -(\mathbf{r_o}\cdot\mathbf{r_o})$:either (±),variable
4-Velocity	$\mathbf{U} = U^\mu = \gamma(c,\mathbf{u})$	$= d\mathbf{R}/d\tau = d^1\mathbf{R}/d\tau^1$	$(\mathbf{U}\cdot\mathbf{U}) = (c)^2$:temporal(+),fundamental constant
4-Acceleration	$\mathbf{A} = A^\mu = \gamma(c\gamma',\gamma'\mathbf{u}+\gamma\mathbf{a})$	$= d\mathbf{U}/d\tau = d^2\mathbf{R}/d\tau^2$	$(\mathbf{A}\cdot\mathbf{A}) = -(a_o)^2 = -(\alpha)^2 = (i\alpha)^2$:spatial(–),variable

An alternate form of the 4-Acceleration handy for circular motion:
4-Acceleration $\mathbf{A} = A^\mu = (\ \gamma^4(\mathbf{a}\cdot\mathbf{u})/c\ ,\ \gamma^4(\mathbf{a}\cdot\mathbf{u})\mathbf{u}/c^2+\gamma^2\mathbf{a}\ ) \ \to\ \gamma^2(0,\mathbf{a})_\perp$ if $(\mathbf{a}\cdot\mathbf{u})=0$

Circular Motion: constants {Radius R, Angular Velocity Ω, Lorentz Factor γ}, which constrain the motion to spatial circular arcs.
4-Position $\mathbf{R} = R^\mu = (ct,\mathbf{r} = R\ \hat{\mathbf{r}} = \mathbf{x}+\mathbf{y}+\mathbf{z}) = (ct,\mathbf{r} = R\cos[\theta]\ \hat{\mathbf{n}}_1 + R\sin[\theta]\ \hat{\mathbf{n}}_2)$ { + const $(ct_{COR},\mathbf{r}_{COR})$ }SpaceTime Translation
4-Velocity $\mathbf{U} = U^\mu = \gamma^1(c,\mathbf{u} = R\Omega\ \hat{\boldsymbol{\theta}})$ $= \gamma(c,\mathbf{u} = -R\Omega\sin[\theta]\ \hat{\mathbf{n}}_1 + R\Omega\cos[\theta]\ \hat{\mathbf{n}}_2)$
4-Acceleration $\mathbf{A} = A^\mu = \gamma^2(0,\mathbf{a} = -R\Omega^2\ \hat{\mathbf{r}})$ $= \gamma^2(0,\mathbf{a} = -R\Omega^2\cos[\theta]\ \hat{\mathbf{n}}_1 - R\Omega^2\sin[\theta]\ \hat{\mathbf{n}}_2)$: Note $|\mathbf{a}| = R\Omega^2$

Hyperbolic Motion: constants {Rindler "Dist" $D=c^2/\alpha$, Proper Accel α}, which constrain the motion to spacetime hyperbolic arcs.
4-Position $\mathbf{R} = R^\mu = (c^2/\alpha)(\sinh[\alpha\tau/c],\cosh[\alpha\tau/c]\ \hat{\mathbf{n}}\ ) = (D)(\sinh[c\tau/D],\cosh[c\tau/D]\ \hat{\mathbf{n}}\ )$ { + const $(ct_{RP},\mathbf{r}_{RP})$ }SpaceTime Translation
4-Velocity $\mathbf{U} = U^\mu = (c)(\cosh[\alpha\tau/c],\sinh[\alpha\tau/c]\ \hat{\mathbf{n}}\ ) = (c)(\cosh[c\tau/D],\sinh[c\tau/D]\ \hat{\mathbf{n}}\ )$
4-Acceleration $\mathbf{A} = A^\mu = (\alpha)(\sinh[\alpha\tau/c],\cosh[\alpha\tau/c]\ \hat{\mathbf{n}}\ ) = (c^2/D)(\sinh[c\tau/D],\cosh[c\tau/D]\ \hat{\mathbf{n}}\ )$: Note $\alpha=c^2/D$
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The general idea of this paper is first to prove that the Invariant Interval for any given pair of points is the same in different frames, for both rotations and boosts. This is typically a passive-transformation: "What does the setup look like from another reference frame?"

Then, since a passive-transformation is equivalent to an active-transformation, we can then create an active-transformation path or trajectory in the initial frame, which also preserves the Invariant Interval along the entire pathways for these two paired points. In essence, we are creating the worldlines. Since the Invariant Interval remains the same for the two paired points in that frame, it will remain invariant in any Poincaré-Transformed frame as well. The "spacetime distance" between the points is a geometric invariant.

For the rotated frame, this is obviously an invariant proper-length, the distance between the two spatial event points. We have everyday experience and good intuition with rotations, so this will later be a guide for boosts.

For the boosted frame, there are actually several different cases. This is due to the causal <Time·Space> structure of the light cone. Event points that are initially space-like separated will remain space-like separated, and will have a real, measurable proper-length. Event points that are time-like separated will remain time-like separated and will have a real, measurable proper-time interval. Event points that are light-like separated will remain light-like separated, and will remain null. The Spaceship Paradox is about space-like separated events. We will preserve proper-length between these points. Proper spatial distances are measured at simultaneous times, meaning when $\Delta t = 0$ in a given frame. When this condition holds, the Δx length in that frame is the proper-length.

If this proper-length is maintained along the entire pathway for one frame, then by the Invariant Interval, it remains so in the other frame, and the string doesn't break under motion for that kind of path or trajectory.

Again, this is easy to visualize for rotations, since we have a lot of everyday experience with those. Then, apply that knowledge to boosts. We will see that the mathematics and physical interpretation behind the two types is amazingly similar.

Scalar Invariance, shown in compact tensor-style, using the SR:Minkowski Metric $\eta_{\mu\nu} = \eta^{\mu\nu} \to$Diagonal$[+1,-1,-1,-1]$(Cartesian coordinates)
$\mathbf{A'}\cdot\mathbf{B'} = A^{\mu'}\eta_{\mu'\nu'}B^{\nu'} = +(a^{0'}b^{0'})-(a^{1'}b^{1'})-(a^{2'}b^{2'})-(a^{3'}b^{3'})$
$(\Lambda^{\mu'}{}_\alpha A^\alpha)\ \eta_{\mu'\nu'}(\Lambda^{\nu'}{}_\beta B^\beta) = (\Lambda^{\mu'}{}_\alpha\eta_{\mu'\nu'}\Lambda^{\nu'}{}_\beta)\ A^\alpha B^\beta = (\Lambda_{\nu'\alpha}\Lambda^{\nu'}{}_\beta)\ A^\alpha B^\beta = (\eta_{\rho\alpha}\Lambda^\rho{}_{\nu'}\Lambda^{\nu'}{}_\beta)\ A^\alpha B^\beta = (\eta_{\alpha\rho}\delta^\rho{}_\beta)\ A^\alpha B^\beta = (\eta_{\alpha\beta})\ A^\alpha B^\beta = A^\alpha(\eta_{\alpha\beta})B^\beta =$
$A^\alpha\eta_{\alpha\beta}B^\beta = \mathbf{A}\cdot\mathbf{B} = +(a^0b^0)-(a^1b^1)-(a^2b^2)-(a^3b^3)$ and hence $\mathbf{A}\cdot\mathbf{A} = +(a^0)^2-(a^1)^2-(a^2)^2-(a^3)^2$
which shows general $\mathbf{\Lambda}$ invariance properties, but now we will show it explicitly in $\mathbf{\Lambda}\to$**R**otation and $\mathbf{\Lambda}\to$**B**oost frames.

Calculations for Rotational Invariance = Proof that Rotation preserves the space-space part of the Invariant Interval:
Let (x_1, y_1) & (x_2, y_2) be any two event points in an initial spatial (x,y)-frame of reference. (t and z remain unchanged)
The Invariant Interval between these points is: $(x_2-x_1)^2+(y_2-y_1)^2=(\Delta x)^2+(\Delta y)^2=R^2$ (R=Radius=Distance to Center-of-Rotation)

[x'] = [cos(θ) -sin(θ)][x] + [d_x] = **Rotation Transformation + Space Translation**
[y'] [sin(θ) cos(θ)][y] + [d_y]

$x'=cx-sy+d_x$: Here use c=cos(θ), s=sin(θ), **d**=spacetime displacement.
$y'=sx+cy+d_y$

$\Delta x' = (x_2'-x_1') = (cx_2-sy_2+d_x)-(cx_1-sy_1+d_x) = c(x_2-x_1)-s(y_2-y_1) = c\Delta x-s\Delta y$:Independent of spacetime translation **d**
$\Delta y' = (y_2'-y_1') = (sx_2+cy_2+d_y)-(sx_1+cy_1+d_y) = s(x_2-x_1)+c(y_2-y_1) = s\Delta x+c\Delta y$

$(\Delta x')^2+(\Delta y')^2$
$=(c\Delta x-s\Delta y)^2+(s\Delta x+c\Delta y)^2$
$=(c^2\Delta x^2-2\,c\Delta x\,s\Delta y+s^2\Delta y^2)+(s^2\Delta x^2+2\,s\Delta x\,c\Delta y+c^2\Delta y^2)$
$=(c^2\Delta x^2+s^2\Delta y^2)+(s^2\Delta x^2+c^2\Delta y^2)$
$=c^2(\Delta x^2+\Delta y^2)+s^2(\Delta x^2+\Delta y^2)$
$=(c^2+s^2)(\Delta x^2+\Delta y^2)$
$=(\Delta x^2+\Delta y^2)$: because $\cos^2+\sin^2=+1$
$=(\Delta x)^2+(\Delta y)^2$: Proving that the Invariant Interval is the same for any two rotational frames, regardless of rotation angle (θ)

Any two points (x_1, y_1) & (x_2, y_2) in one frame of reference have an Invariant Interval $(\Delta x)^2+(\Delta y)^2=R^2$
The same two points (x_1', y_1') & (x_2', y_2') in a Poincaré-rotated frame of reference have the same Invariant Interval $(\Delta x')^2+(\Delta y')^2=R^2$

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Calculations for Boost Invariance = Proof that Boost preserves the time-space part of the Invariant Interval:
Let (ct_1, x_1) & (ct_2, x_2) be any two event points in an initial time:space (t,x)-frame of reference. (y and z remain unchanged)
The Invariant Interval between these points is: $(ct_2-ct_1)^2-(x_2-x_1)^2=(c\Delta t)^2-(\Delta x)^2=D^2$ (D=Distance to Rindler Point:Center-of-Hyperbola)

[ct'] = [cosh(φ) sinh(φ)][ct] + [d_t] = **Boost Transformation + Space Translation**
[x'] [sinh(φ) cosh(φ)][x] + [d_x]

Temporarily setting light-speed (c)=1 so {ct →t} to not confuse with cosh(φ).

$t'=ct+sx+d_t$: Here use c=cosh(φ), s=sinh(φ), **d**=spacetime displacement.
$x'=st+cx+d_x$

$\Delta t' = (t_2'-t_1') = (ct_2+sx_2+d_t)-(ct_1+sx_1+d_t) = c(t_2-t_1)+s(x_2-x_1) = c\Delta t+s\Delta x$:Independent of spacetime translation **d**
$\Delta x' = (x_2'-x_1') = (st_2+cx_2+d_x)-(st_1+cx_1+d_x) = s(t_2-t_1)+c(x_2-x_1) = s\Delta t+c\Delta x$

$(\Delta t')^2-(\Delta x')^2$
$=(c\Delta t+s\Delta x)^2-(s\Delta t+c\Delta x)^2$
$=(c^2\Delta t^2+2\,c\Delta t\,s\Delta x+s^2\Delta x^2)-(s^2\Delta t^2+2\,s\Delta t\,c\Delta x+c^2\Delta x^2)$
$=(c^2\Delta t^2+s^2\Delta x^2)-(s^2\Delta t^2+c^2\Delta x^2)$
$=c^2(\Delta t^2-\Delta x^2)-s^2(\Delta t^2-\Delta x^2)$
$=(c^2-s^2)(\Delta t^2-\Delta x^2)$
$=(\Delta t^2-\Delta x^2)$: because $\cosh^2-\sinh^2=+1$
$=(\Delta t)^2-(\Delta x)^2$

Return light-speed value to symbol (c) so {t →ct}, no more possible confusion since cosh(φ) is gone
$=(c\Delta t)^2-(\Delta x)^2$: Proving that the Invariant Interval is the same for any two boosted frames, regardless of hyperangle (φ)

Any two event points (ct_1, x_1) & (ct_2, x_2) in one frame reference have an Invariant Interval $(c\Delta t)^2-(\Delta x)^2=D^2$
The same two event points in a Poincaré-boosted frame of reference have the same Invariant Interval $(c\Delta t')^2-(\Delta x')^2=D^2$

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This is actually all general from the full tensor form: $\mathbf{A'}\cdot\mathbf{B'} = A^{\mu'}\eta_{\mu'\nu'}B^{\nu'}$ = (dummy indexes may be changed) = $A^{\alpha}\eta_{\alpha\beta}B^{\beta} = \mathbf{A}\cdot\mathbf{B}$
but showing it for subsets of **R**otation and **B**oost clarifies things, showing they are both partial aspects of the same thing.
Full Invariant Interval: $+(ct')^2 - (x')^2 - (y')^2 - (z')^2 = +(ct)^2 - (x)^2 - (y)^2 - (z)^2 = \mathbf{R'}\cdot\mathbf{R'} = \mathbf{R}\cdot\mathbf{R}$
Full Invariant Interval: $+(c\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2 = +(c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 = \Delta\mathbf{R'}\cdot\Delta\mathbf{R'} = \Delta\mathbf{R}\cdot\Delta\mathbf{R}$

A very important aspect of these Lorentz and Poincaré transformations is the idea of active and passive transforms. An active transform is one in which the objects being measured are physically acted upon. They are accelerated and moved, requiring work. A passive transform is one in which the observer moves (or simply rearranges their measuring system, as all linearly-connected measuring systems are equally valid) and there is no acceleration or motion or work done on the object being measured.

The interesting point here is that a given active and related passive transform are mathematically identically-equal inverses of one another. An affine transform (or change of basis) is one that preserves the Invariant Interval. Since a passive transform applies to a frame as a whole (the observer moves without affecting the system under observation), its inverse is physically natural as well. One can move an object, thought of as a Displacement 4-Vector, or 4-Displacement, in such a way that the Invariant Interval is unchanged.
https://en.wikipedia.org/wiki/Active_and_passive_transformation
https://en.wikipedia.org/wiki/Change_of_basis

I wrote an online simulator that allows one to try all of these transformation types. It confirms Invariant Interval preservation.
http://scirealm.org/Physics-MinkowskiDiagram.html

Another *critical* point is that the transformations are perceptual, both for rotations and boosts. This is obvious for a passive transformation. Literally, nothing is done to the object. For an active transformation, to be equivalent to the passive, the object must be moved in a special way such that the Invariant Interval is unchanged. This can be accomplished, as shown, actually in a couple of different ways. This concept also matches the idea that components of 4-Vectors can vary with frame. They are relative to frame↔perceptual. It is the Invariant Intervals which are absolute. Again, Tensor Invariants = {Inner Products, Traces, Determinants}.

An easy example is rotating a stick about the (x,y)-plane. Let the stick initially lie entirely along the x-axis. Now rotate it a small amount. The component value along x, known as the projection-along-x, is now smaller, and the component value along y, known as the projection-along-y, is now larger. Continue until the stick is entirely along the y-axis. There is nothing mysterious here. It is standard vector-projection. The length of the actual stick is unchanged. The invariant rule is simply that $\text{length}^2 = \Delta x^2+\Delta y^2$. When the stick is along the x-axis, $\Delta y=0$, $\Delta x=\text{length}$. When the stick is along the y-axis, $\Delta x=0$, $\Delta y=\text{length}$. The frame x and y values are perceptual only. They don't "do" anything to the proper length of the stick. Measured values (x,y) are relative to the observer, but in such a way that system invariants are preserved ($\Delta x^2+\Delta y^2 = \text{invariant}$). Likewise, one may instead rotate the coordinate system, such that the x-axis becomes the y'-axis, the y-axis becomes the negative x'-axis, etc. Again, nothing physically happens to the stick.

The boost case seems more complicated, but it really isn't that much different. It's just a <Time·Space> 4D "rotation" about the (t,x)-plane. Let's use an actual observed case, that of muons coming into the atmosphere. Muons created in the lab are short half-lived and decay quickly. Normally, such muons would not live long enough to come through the atmosphere and be detected at ground level, even at their high velocities. However, extra-planetary muons typically move at relativistic speeds, so neat stuff happens. In the Earth-frame, the muons are time-dilated, so "live longer" and are able to reach the ground. In the muon-frame, they see the atmosphere as length-contracted, so they don't have to travel as far and according to their half-life clocks, which tick normally at the usual rate in their own frames, and it enables them to reach the ground.

Note the following however: There is nothing physical about the atmosphere that should change the physical half-life constant of a muon. Likewise, there is nothing physical about a muon that should physically shrink the atmosphere. These effects are "perceptual". The muon's half-life clock ticks normally in its own frame. The atmosphere maintains its height in its own frame. The invariant rule is simply that $\text{interval}^2=(c\Delta t)^2-\Delta x^2$. This is totally analogous to the rotation rule of $\text{length}^2 = \Delta x^2+\Delta y^2$. In exactly the same way that a stick can trade (x) & (y), which physically does nothing to the stick, the muon/atmosphere path can trade (ct) & (x), which physically does nothing to the muon nor atmosphere.

Can these transformations then still have physical consequences? Yes, absolutely! Try to putting a key in a keyhole sideways. It doesn't go. Rotate the key so that it is aligned correctly. It can now go in the keyhole. Rotating the key did nothing to the key or keyhole physically, it was just a dimensional-alignment. Same thing with muons. Send one down through the atmosphere at slow or even classically high speed. It doesn't reach the ground as a muon, it decays too soon. But, boost the muon to relativistic speed. It can now reach the ground before decaying. Boosting does not physically change the muon's internal half-life rest tick-rate. Boosting does not physically shrink the atmosphere in its rest frame. Boosting is just an alignment of dimensional axes, just like with the key.

Now, the whole string-snapping aspect of the Bell Spaceship Paradox would be an invariant event. If it happens in one frame, it must happen in the others. If it doesn't happen in one frame, then it doesn't happen in any other frames. If the string does snap, it must be based on a force or stress. This can be caused by an acceleration difference between two events that does not preserve the Invariant Interval. There is no stress caused by a passive Poincaré transform, nor by an equivalent active Poincaré transform, both of which preserve the Invariant Interval. So, we will examine under what circumstances the Intervals between the points are or are not preserved. Note for later: there can also be a snap due to stress from the overall magnitude of the acceleration, but this is not related to the transforms.

Now, we will create the active-transform trajectories/paths that also preserve the Invariant Intervals along those paths.

Calculations for Active-Transform Rotation Paths:
Pick a totally arbitrary initial starting point (x_o , y_o) and place the Center-of-Rotation (COR) points and ships arbitrarily.
Then, we want the two points to move in such a way as to maintain invariant distance regardless of a rotation angle (θ).
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Calculate for event points using Independent Centers-of-Rotation (COR_1 & COR_2).
The coordinates of the two points in a lab reference frame are:
$(x_1 , y_1) = (x_o+x_{COR1}+r_1 \cos[\theta_1] , y_o+y_{COR1}+r_1 \sin[\theta_1])$
$(x_2 , y_2) = (x_o+x_{COR2}+r_2 \cos[\theta_2] , y_o+y_{COR2}+r_2 \sin[\theta_2])$

The displacement between the points is:
$(\Delta x , \Delta y)$
$= (x_2-x_1 , y_2-y_1)$
$= (x_o+x_{COR2}+r_2 \cos[\theta_2]-x_o-x_{COR1}-r_1 \cos[\theta_1] , y_o+y_{COR2}+r_2 \sin[\theta_2]-y_o-y_{COR1}-r_1 \sin[\theta_1])$
$= (x_{COR2}-x_{COR1}+r_2 \cos[\theta_2]-r_1 \cos[\theta_1] , y_{COR2}-y_{COR1}+r_2 \sin[\theta_2]-r_1 \sin[\theta_1])$
$= (\Delta x_{COR}+r_2 \cos[\theta_2]-r_1 \cos[\theta_1] , \Delta y_{COR}+r_2 \sin[\theta_2]-r_1 \sin[\theta_1])$ i.e. the initial reference point (x_o , y_o) doesn't matter

$Dist^2$ = {4D Invariant}
$= (\Delta x^2+\Delta y^2) = (\Delta x'^2+\Delta y'^2)$
$=\{\Delta x_{COR}+r_2 \cos[\theta_2]-r_1 \cos[\theta_1]\}^2+\{\Delta y_{COR}+r_2 \sin[\theta_2]-r_1 \sin[\theta_1]\}^2$

$=r_1^2+r_2^2-2r_1r_2\cos(\theta_1-\theta_2)+2r_1\{-\Delta x_{COR}\cos(\theta_1)-\Delta y_{COR}\sin(\theta_1)\}+2r_2\{\Delta x_{COR}\cos(\theta_2)+\Delta y_{COR}\sin(\theta_2)\}+\Delta x_{COR}^2+\Delta y_{COR}^2$
Let both models rotate by angle θ, but have different initial phases $\{\theta_1=\theta+p_1 ; \theta_2=\theta+p_2\}$

$=r_1^2+r_2^2-2r_1r_2\cos(p_1-p_2)+2r_1\{-\Delta x_{COR}\cos(\theta+p_1)-\Delta y_{COR}\sin(\theta+p_1)\}+2r_2\{\Delta x_{COR}\cos(\theta+p_2)+\Delta y_{COR}\sin(\theta+p_2)\}+\Delta x_{COR}^2+\Delta y_{COR}^2$
which is not generally independent of the rotation angle (θ).

Now, how can we make this independent of the path parameter, i.e. of rotation angle (θ)?

The 1st way:
Choose a Common Center of Rotation (COR), by setting $\Delta x_{COR}=\Delta y_{COR}=0$.
Giving $(\Delta x , \Delta y) = (r_2 \cos[\theta+p_2]-r_1 \cos[\theta+p_1] , r_2 \sin[\theta+p_2]-r_1 \sin[\theta+p_1])$
Then: $Dist^2 = r_1^2+r_2^2-2r_1r_2\cos(p_1-p_2)$, which is independent of rotation angle (θ)
This is like setting models on a turn-table or Lazy-Susan; initial phases and radii don't matter, other than setting the initial Invariant.
It's a standard Homogeneous Lorentz Rotation. $Dist^2 = (r_1-r_2)^2$ if phases aligned(0°) or $(r_1+r_2)^2$ if phases anti-aligned (180°).

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The 2nd way:
Use Independent Centers-of-Rotation (COR_1 & COR_2).
Choose equal Radii, by setting $r_2=r_1$: This equates to having equal acceleration magnitudes by $|\mathbf{a}| = R\Omega^2$, since Ω from equal θ's.
$=r_1^2+r_2^2-2r_1r_2\cos(p_1-p_2)+2r_1\{-\Delta x_{COR}\cos(\theta+p_1)-\Delta y_{COR}\sin(\theta+p_1)\}+2r_2\{\Delta x_{COR}\cos(\theta+p_2)+\Delta y_{COR}\sin(\theta+p_2)\}+\Delta x_{COR}^2+\Delta y_{COR}^2$
$=2r_1^2-2r_1^2\cos(p_1-p_2)+2r_1[-\Delta x_{COR}\cos(\theta+p_1)-\Delta y_{COR}\sin(\theta+p_1)+\Delta x_{COR}\cos(\theta+p_2)+\Delta y_{COR}\sin(\theta+p_2)]+\Delta x_{COR}^2+\Delta y_{COR}^2$
&
Choose equal phases: by setting $p_1=p_2$: This means the starting angular displacements are the same at each COR.
$= 2r_1^2-2r_1^2\cos(p_1-p_1)+2r_1[-\Delta x_{COR}\cos(\theta+p_1)-\Delta y_{COR}\sin(\theta+p_1)+\Delta x_{COR}\cos(\theta+p_1)+\Delta y_{COR}\sin(\theta+p_1)]+\Delta x_{COR}^2+\Delta y_{COR}^2$
$= 2r_1^2-2r_1^2\cos(0)+2r_1[0]+\Delta x_{COR}^2+\Delta y_{COR}^2$
Then: $Dist^2 = \Delta x_{COR}^2+\Delta y_{COR}^2$, which is independent of rotation angle (θ)
This is like the endpoints of two equivalently-rotating, same-length dials, with same radii and phase.
It's a Poincaré Rotation, or Inhomogeneous Lorentz Rotation, with Lorentz Rotation + Space Translation.
In fact, $(\Delta x , \Delta y) = (x_2-x_1 , y_2-y_1) = (\Delta x_{COR} , \Delta y_{COR})$, or $(x_2 , y_2) = (x_1 , y_1)+(\Delta x_{COR} , \Delta y_{COR})$
The 2nd point exactly follows the 1st point. It's Invariant under the SpaceTime Translation part of Poincaré Invariance.

Both of these types of motions (the 2 ways) preserve the Invariant Interval (ie. proper-lengths) along their entire trajectories.

Calculations for Active-Transform Boost Paths/Worldlines:
Pick a totally arbitrary initial starting point (ct_o , x_o) and place the Rindler Points (RP)=hyperbolic (COR) and ships arbitrarily.
Then, we want the two points to move in such a way as to maintain interval regardless of a hyperbolic-rotation/boost angle (φ).

Calculate for event points using Independent Rindler Points (RP_1 & RP_2):
The coordinates of the two points in a lab reference frame are:
$(ct_1 , x_1) = (ct_o+ct_{RP1}+d_1 \sinh[\varphi_1] , x_o+x_{RP1}+d_1 \cosh[\varphi_1])$
$(ct_2 , x_2) = (ct_o+ct_{RP2}+d_2 \sinh[\varphi_2] , x_o+x_{RP2}+d_2 \cosh[\varphi_2])$

The displacement between the points is:
$(c\Delta t , \Delta x)$
$= (ct_2-ct_1 , x_2-x_1)$
$= (ct_o+ct_{RP2}+d_2 \sinh[\varphi_2]-ct_o-ct_{RP1}-d_1 \sinh[\varphi_1] , x_o+x_{RP2}+d_2 \cosh[\varphi_2]-x_o-x_{RP1}-d_1 \cosh[\varphi_1])$
$= (ct_{RP2}-ct_{RP1}+d_2 \sinh[\varphi_2]-d_1 \sinh[\varphi_1] , x_{RP2}-x_{RP1}+d_2 \cosh[\varphi_2]-d_1 \cosh[\varphi_1])$
$= (c\Delta t_{RP}+d_2 \sinh[\varphi_2]-d_1 \sinh[\varphi_1] , \Delta x_{RP}+d_2 \cosh[\varphi_2]-d_1 \cosh[\varphi_1])$ i.e. the initial reference point (ct_o , x_o) doesn't matter

IntervalDist^2 {4D Invariant}
$= (c^2\Delta t^2-\Delta x^2) = (c^2\Delta t'^2-\Delta x'^2)$
$=\{c\Delta t_{RP}+d_2 \sinh[\varphi_2]-d_1 \sinh[\varphi_1]\}^2-\{\Delta x_{RP}+d_2 \cosh[\varphi_2]-d_1 \cosh[\varphi_1]\}^2$

$=-d_1^2-d_2^2+2d_1d_2\cosh(\varphi_1-\varphi_2)+2d_1\{-c\Delta t_{RP} \sinh(\varphi_1)+\Delta x_{RP}\cosh(\varphi_1)\}+2d_2\{c\Delta t_{RP}\sinh(\varphi_2)-\Delta x_{RP}\cosh(\varphi_2)\}+c\Delta t_{RP}^2-\Delta x_{RP}^2$
Let both models rotate by hyperangle φ, but have different initial phases $\{\varphi_1=\varphi+p_1 ; \varphi_2=\varphi+p_2\}$
$=-d_1^2-d_2^2+2d_1d_2\cosh(p_1-p_2)+2d_1\{-c\Delta t_{RP} \sinh(\varphi+p_1)+\Delta x_{RP}\cosh(\varphi+p_1)\}+2d_2\{c\Delta t_{RP}\sinh(\varphi+p_2)-\Delta x_{RP}\cosh(\varphi+p_2)\}+c\Delta t_{RP}^2-\Delta x_{RP}^2$

which is not generally independent of the hyperbolic rotation angle (φ). Note the similarity to the rotation equations.

Now, how can we make this independent of the path parameter, i.e. of hyperbolic-rotation angle (φ)?

The 1st way:
Choose a Common Rindler Point (RP), by setting $\Delta t_{RP}=\Delta x_{RP}=0$.
Giving $(c\Delta t , \Delta x) = (d_2 \sinh[\varphi+p_2]-d_1 \sinh[\varphi+p_1] , d_2 \cosh[\varphi+p_2]-d_1 \cosh[\varphi+p_1])$
Then: IntervalDist^2 $= -d_1^2-d_2^2+2d_1d_2\cosh(p_1-p_2)$, which is independent of the boost hyperbolic-rotation angle (φ)
This is pure hyperbolic acceleration of two endpoints, with one accelerating harder, initial Rindler Distances and phases don't matter other than setting the initial invariant.
It's a standard Homogeneous Lorentz Boost. IntervalDist^2 $= -(d_1-d_2)^2$ ={spatial} if phases aligned. There is not an anti-aligned case.
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The 2nd way:
Use Independent Rindler Points (RP_1 & RP_2).
Choose equal Rindler Distances, by setting $d_1=d_2$: This equates to having equal proper-accelerations by $\alpha = c^2/D$
$=-d_1^2-d_1^2+2d_1d_1\cosh(p_1-p_2)+2d_1\{-c\Delta t_{RP} \sinh(\varphi+p_1)+\Delta x_{RP}\cosh(\varphi+p_1)\}+2d_1\{c\Delta t_{RP}\sinh(\varphi+p_2)-\Delta x_{RP}\cosh(\varphi+p_2)\}+c\Delta t_{RP}^2-\Delta x_{RP}^2$
$=-2d_1^2+2d_1^2\cosh(p_1-p_2)+2d_1\{-c\Delta t_{RP} \sinh(\varphi+p_1)+\Delta x_{RP}\cosh(\varphi+p_1)+c\Delta t_{RP}\sinh(\varphi+p_2)-\Delta x_{RP}\cosh(\varphi+p_2)\}+c\Delta t_{RP}^2-\Delta x_{RP}^2$
&
Choose equal phases: by setting $p_1=p_2$: This equates to the thrust start times matching Rindler Point times in the start frame.
$=-2d_1^2+2d_1^2\cosh(p_1-p_1)+2d_1\{-c\Delta t_{RP} \sinh(\varphi+p_1)+\Delta x_{RP}\cosh(\varphi+p_1)+c\Delta t_{RP}\sinh(\varphi+p_1)-\Delta x_{RP}\cosh(\varphi+p_1)\}+c\Delta t_{RP}^2-\Delta x_{RP}^2$
$=-2d_1^2+2d_1^2\cosh(0)+c\Delta t_{RP}^2-\Delta x_{RP}^2$
Then: IntervalDist^2 $= c\Delta t_{RP}^2 - \Delta x_{RP}^2$, which is independent of the boost hyperbolic-rotation angle (φ)
If one then chooses $c\Delta t_{RP} = 0$ = Identical Start Times, then IntervalDist^2 $= - \Delta x_{RP}^2$
This is the setup of the Bell Spaceship Paradox, two ships with equal accelerations & begin boost at same time in start frame.
It's a Poincaré Boost = Inhomogeneous Lorentz Boost = Lorentz Boost + SpaceTime Translation.
In fact, $(c\Delta t , \Delta x) = (ct_2-ct_1 , x_2-x_1) = (c\Delta t_{RP} , \Delta x_{RP})$, or $(ct_2,x_2) = (ct_1,x_1)+(c\Delta t_{RP} , \Delta x_{RP})$
As with Rotation, The 2nd follows the 1st. It's Invariant under the SpaceTime Translation part of Poincaré Invariance.

Both of these types of motions (the 2 ways) preserve the Invariant Interval (ie. proper-lengths since they are spatial) along their entire trajectories.

$\varphi_1 = \alpha_1\tau_1/c = c\tau_1/d_1 = \varphi+p_1$ & $\varphi_2 = \alpha_2\tau_2/c = c\tau_2/d_2 = \varphi+p_2$

Note the similarity of Boost Results to Rotation Results: They are isomorphic, with just a sign difference from the Minkowski Metric:
{cos↔cosh **:** sin↔sinh **:** Radius R↔RindlerDistance D **:** Center of Rotation↔RindlerPoint=Center of Hyperbolic "Rotation"}
{3-vector magnitudes $|\mathbf{a}| = |\mathbf{u}|^2/|\mathbf{r}| = (R\Omega)^2/R = R\Omega^2$ ↔ 4-Vector Magnitudes $|\mathbf{A}| = |\mathbf{U}|^2/|\mathbf{R}| = \alpha = c^2/D$}

Consider the same situation as set-up in the paradox, but with two model ships in a laboratory on a turn-table, or Lazy-Susan. The two ships are of the same length, lined up one behind another, connected by a taut string. They share a common Center of Rotation.

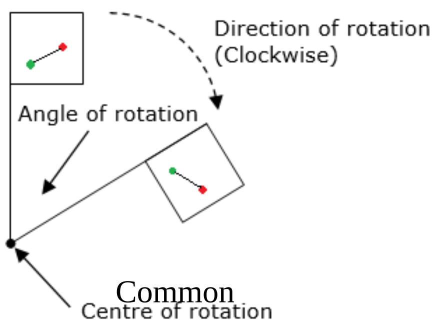
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$ = **Rotation Transformation (Homogeneous)**

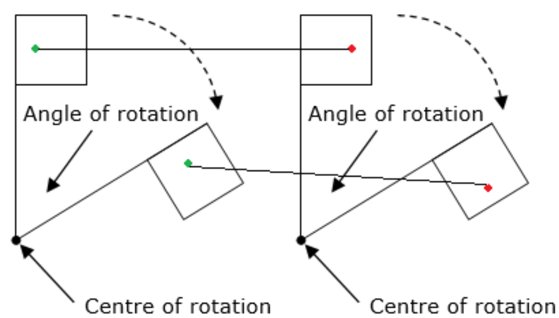
In the 1st scenario, rotate the turn-table by some angle (θ). The two ships will both have their headings rotated by (θ) degrees, the distances are trivially preserved in the rotated frame, preserving the invariances, and the string remains attached.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$ = **Rotation Transformation + Space Translation (Inhomogeneous)**

In the 2nd scenario, we lock the turn-table so that it can't turn. Instead, rotate each ship individually by (θ) degrees about independent Centers of Rotation (COR). In this case, the ships will both still have their headings rotated by (θ) degrees, but they are no longer aligned head to tail (instead are somewhat sideways to one another), the distances between the ships are not generally preserved, and the string connecting them can break.

What is happening? In both cases, the ships have rotated by the same angle (θ).





However, in the 1st scenario, the two ships have a Common Center of Rotation (COR). They share a common reference point in both the initial and final (rotated) frame. The distances between all points of the ships is maintained. Invariance is maintained. This is standard Lorentz Rotation.

In the 2nd scenario, even though they rotate by the same amount (θ), they are using their own independent Centers of Rotation (COR_1 & COR_2). The placements do not in general preserve the distances, and the string loosens or breaks for those cases. Invariance is not generally maintained. There is, however, a single special case where the distances are maintained, and that is when the radii match and phases match (same start angle) This is a Poincaré Rotation. It preserves the Invariant Interval. This motion does NOT break the string.

An important point here is that in just regular standard rotation, the points that are further (R) from the Center of Rotation must accelerate harder ($|\mathbf{a}| = |\mathbf{u}|^2/|\mathbf{r}| = (R\Omega)^2/R = R\Omega^2$) to maintain coordinated distances as the object is physically rotated (Ω). The reason I mention this is that it becomes important in understanding what happens in the boost.

Also, it is important to note that the x-components and y-components as seen in any frame don't physically "do" anything to the ships. They are the *relative* internal components of the 4-Vector measurements, which by definition vary by frame. 4-Vector components are relative, not absolute. The *absolute* thing is the Invariant Interval, which is the geometric inner product of 4-Vectors.

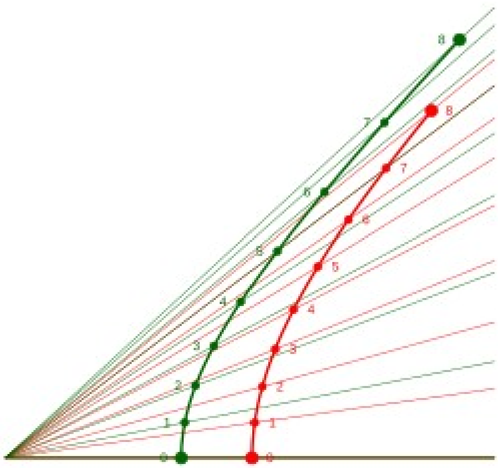
If you look at an object from the front, eg. x-axis, then you look at the same object from the side, eg. y-axis, you haven't physically altered the object itself. You are simply changing a frame of reference. The only thing that can "alter" the object is differential accelerations, which cause forces/stresses in the object. Length alterations from x-projection and y-projection in rotation are perceptual only.

Now, let's examine the original setup, with the real ships boosting in space.

[ct'] = [cosh(φ) -sinh(φ)][ct] = [γ -γβ][ct] = **Boost Transformation (Homogeneous)**
[x'] [-sinh(φ) cosh(φ)][x] = [-γβ γ][x]

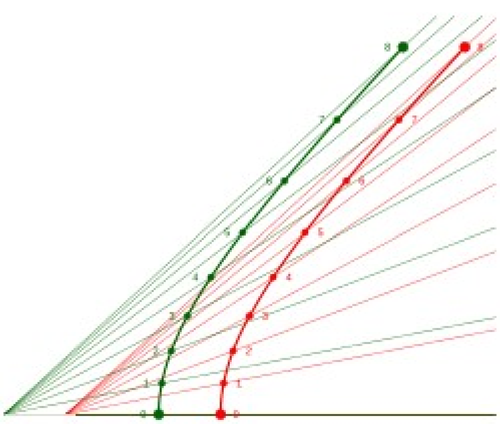
In both cases, the ships will "hyperbolically rotate", or boost, by the same spacetime "hyperbolic angle" ($\varphi = \alpha\tau/c = c\tau/D$), just like the spatial rotations using the same spatial angle (θ).

In the 1st scenario, let the ships boost with a common frame, a proper hyperbolic acceleration, in which they share the same Rindler Point (RP), which is like a hyperbolic Center of Rotation. In this case, the distances in the ship's boosted reference frame are preserved, the invariances are maintained, and the string doesn't break. As it turns out, in this scenario, they do NOT use identical start-frame acceleration profiles. The ship in back, closer to the Rindler Point, must accelerate harder ($|\mathbf{A}| = |\mathbf{U}|^2/|\mathbf{R}| = \alpha = c^2/D$), in the same way that a rotated point further from the COR must accelerate harder. Note that in the start frame the ships seem to be getting closer, from Lorentz length contraction, but they are actually maintaining the proper start distance in the ship frame-of-reference. This is a regular Lorentz Boost.



[ct'] = [cosh(φ) -sinh(φ)][ct] + [cΔt] = [γ -γβ][ct] + [cΔt] = **Boost Transformation + SpaceTime Translation (Inhomogeneous)**
[x'] [-sinh(φ) cosh(φ)][x] + [Δx] = [-γβ γ][x] + [Δx]

In the 2nd scenario, the ships boost with identical acceleration profiles in the start frame. However, this means that they are each using their own separate, non-identical, Rindler Points {RP_1 & RP_2}. Usually, this would mean that distances between the ships are not preserved in the ship reference frames, and the string loosens or snaps, in exactly the same way as that of rotations using their own separate, non-identical Centers of Rotation usually do. However, as in the case of rotations, there is a special case that preserves the Invariant Interval, and that is when the Rindler Distances match (same accelerations) and the phases match (ie. that use the same start time). The string also does NOT break in this hyperbolic case, which is the Bell's Spaceship setup. Note that the Rindler Distances (D) in this case for each ship are equal, they are just using different Rindler Points. This is a Poincaré Boost. Now, if the Rindler Distances were different, or if the phases were different, then Invariant Interval would not be preserved, and the string would snap.



If you look at an object from a resting inertial frame, then you look at the same object from a moving inertial frame, you haven't physically altered the object itself. You are simply changing a frame of reference. The only thing that can "alter" the object is differential accelerations, which cause forces/stresses in the object. Boosted length-contractions (eg. x-axis) and time-dilations (t-axis) are "perceptual" only, as are the rotated spatial-vector-projections (length-alterations) along x and y. All of these transforms do not, in and of themselves, cause accelerations, but are the result of special types of accelerations, those which preserve the Invariant Interval.

So, the spaceship paradox is resolved. Pure hyperbolic boosts (Common Rindler Points = Lorentz Boost) preserve the Invariant Interval, and the special case of matching Rindler Distances with matching phases on differing Rindler Points (Poincaré Boost) also preserve the Invariant Interval, and thus the string doesn't snap. Any conditions of acceleration other than these loosen or break the string, as the ships will change spatial proper-distance from one another.

Similarly, pure common rotations (Common Centers of Rotation = Lorentz Rotation) preserve the Invariant Interval, and the special case of matching Radii with matching phases on differing Centers of Rotation (Poincaré Rotation) also preserve the Invariant Interval. Any conditions of acceleration other than these loosen or break the string, as the ships change proper-distance from one another.

We note the similarities in the two types of Transforms.
Rotations have $(x)^2 + (y)^2 = R^2$, with R being a constant distance to a Center of Rotation (COR).
Boosts have $(ct)^2 - (x)^2 = D^2$, with D being a constant distance to a Rindler Point (RP), which in this case is also an event horizon.

Rotations have circular paths in space, eg. $\cos(\theta)$ & $\sin(\theta)$ on x and y.
Boosts have hyperbolic paths in spacetime, eg. $\cosh(\varphi)$ and $\sinh(\varphi)$ on t and x.

One important part of this analysis was to use a Poincaré transformation as opposed to just a Lorentz transformation. The Poincaré Transformation includes the Lorentz Transformation plus a Spacetime Translation. It is this extra spacetime translation which leads to the special case of matching Rindler Distances and phases, by allowing the use of different Rindler Points.

If using just Lorentz transformations, the only solution would be using a common Rindler Point. But, the Invariant Interval applies to the full set of the Poincaré transforms. The translation provides that the Rindler Points can be different, but still get a preserved Invariant Interval if the Rindler Distances and phases match.

Length contractions and time dilations are real, but only in a perceptual way, in the same way that vector-projection along an axis is. Can they have physical effects? Sure, as in extra-planetary muons being able to reach the ground. But, the atmosphere has no physical effect on the half-life clocks of muons, and the muons have no ability to physically alter the length/height of our atmosphere. The physical effect is that muons that would not otherwise be able to reach the ground, do, and rotated sticks that won't fit into a hole sideways will fit tip-on.
The Poincaré transformations do exactly that and no more: They re-orient objects along their dimensional axes.

So, we set all this up by creating the equations of motion for the start frame. We then made sure that the general trajectories were those that give circular motion with rotational parameter (angle θ), or hyperbolic motion with boost parameter (hyperbolic angle φ). We then found what constraints must be put on all other initial parameters to maintain the Invariant Interval. If the Invariant Interval is fixed in one frame along the entire trajectory, then it is fixed in all frames. The trivial case for both is using (Common Centers of Rotation:Common Rindler Points). However, there is also a special case for both that still works for independent centers, and that is when the Radii match & phases match for rotations, and when the Rindler Distances match & phases match for boosts. Other cases generally do not preserves distances, and so would loosen or break the string.

An interesting point is that you can also apply a regular turn-table rotation on top of the separate dial version, and it again preserves the invariant interval. Likewise, you could apply a regular hyperbolic boost on top of the separate parallel acceleration setup, and it would again preserve the invariant interval.

Rotations have circular paths in space, eg. x and y.	**B**oosts have hyperbolic paths in spacetime, eg. x and t.
Lorentz Transformation **Λ→R**otation (**R**)otation = All Spatial: 3-vectors { $\lvert\mathbf{r}\rvert,\lvert\mathbf{u}\rvert,\lvert\mathbf{a}\rvert,\lvert\mathbf{j}\rvert$ } & { R,Ω,γ } all constants $\mathbf{a} = (-\Omega^2)\mathbf{r}$, $\mathbf{j} = (-\Omega^2)\mathbf{u}$ $(\mathbf{a}\cdot\mathbf{u}) = 0$, $\gamma' = 0$: 3-acceleration ⊥ 3-velocity $\mathbf{n}_1 = \cos$: $\mathbf{n}_2 = \sin$	Lorentz Transformation **Λ→B**oost (**B**)oost = Time·Space: 4-Vectors { $\lvert\mathbf{R}\rvert,\lvert\mathbf{U}\rvert,\lvert\mathbf{A}\rvert,\lvert\mathbf{J}\rvert$ } & { D,c,α } all constants $\mathbf{A} = (\alpha^2/c^2)\mathbf{R}$, $\mathbf{J} = (\alpha^2/c^2)\mathbf{U}$ $(\mathbf{A}\cdot\mathbf{U}) = 0$: 4-Acceleration ⊥ 4-Velocity $\gamma = \cosh$: $\gamma\beta = \sinh$: $\beta = \tanh$
Circular Case: $-(\mathbf{a}\cdot\mathbf{r}) = (R\Omega)^2 = (\mathbf{u}\cdot\mathbf{u})$ $\lvert\mathbf{a}\rvert = \lvert\mathbf{u}\rvert^2/\lvert\mathbf{r}\rvert = (R\Omega)^2/R = R\Omega^2$ R is fixed distance to Center of Rotation COR	Hyperbolic Case: $-(\mathbf{A}\cdot\mathbf{R}) = (c)^2 = (\mathbf{U}\cdot\mathbf{U})$ $\lvert\mathbf{A}\rvert = \lvert\mathbf{U}\rvert^2/\lvert\mathbf{R}\rvert = \alpha = c^2/D$ D is fixed distance to Rindler Point RP (Hyp. Center of Rotation)
Circular Motion: constants {R,Ω,γ} 4-Position $\mathbf{R} = R^\mu = (ct, \mathbf{r} = R\,\hat{\mathbf{r}})$ 4-Velocity $\mathbf{U} = U^\mu = \gamma^1(c, \mathbf{u} = R\Omega\,\hat{\boldsymbol{\theta}})$ 4-Acceleration $\mathbf{A} = A^\mu = \gamma^2(0, \mathbf{a} = -R\Omega^2\,\hat{\mathbf{r}})$	Hyperbolic Motion: constants {Rindler "Dist" D=c²/α} 4-Position $\mathbf{R} = R^\mu = (c^2/\alpha)(\sinh[\alpha\tau/c], \cosh[\alpha\tau/c]\,\hat{\mathbf{n}})$ 4-Velocity $\mathbf{U} = U^\mu = (c)(\cosh[\alpha\tau/c], \sinh[\alpha\tau/c]\,\hat{\mathbf{n}})$ 4-Acceleration $\mathbf{A} = A^\mu = (\alpha)(\sinh[\alpha\tau/c], \cosh[\alpha\tau/c]\,\hat{\mathbf{n}})$
For an (x,y)-rotation, we have **Λ→R**ot, with Det[**Λ**] = +1: [x'] = [cos(θ) -sin(θ)][x] [y'] [sin(θ) cos(θ)][y] Det[**Λ**] = $\cos^2 + \sin^2 = +1$	For a (t,x)-boost, we have **Λ→B**oost, with Det[**Λ**] = +1: [ct'] = [cosh(φ) -sinh(φ)][ct] = [γ -γβ][ct] [x'] [-sinh(φ) cosh(φ)][x] = [-γβ γ][x] Det[**Λ**] = $\cosh^2 - \sinh^2 = +1$: $\gamma^2 - (\gamma\beta)^2 = +1$: from $\gamma = 1/\mathrm{Sqrt}[1-\beta^2]$
Inner Product Rotation (x,y)-Invariance: $(x')^2 + (y')^2 = (x)^2 + (y)^2 = R^2$: because $\cos^2 + \sin^2 = +1$	Inner Product Boost (t,x)-Invariance: $(ct')^2 - (x')^2 = (ct)^2 - (x)^2 = D^2$: because $\cosh^2 - \sinh^2 = +1$
Generic 4D Rotation Matrix **Λ→R**ot (time, and 3-space) [1 0_j] [0^i $(\delta^i{}_j - n^i n_j)\cos(\theta) - (\varepsilon^i{}_{jk} n^k)\sin(\theta) + n^i n_j$]	Generic 4D Boost Matrix **Λ→B**oost (time, and 3-space) [γ $-\gamma\beta_j$] [$-\gamma\beta^i$ $(\gamma-1)\beta^i\beta_j/(\boldsymbol{\beta}\cdot\boldsymbol{\beta}) + \delta^i{}_j$]
Trace[**Λ**] = {0..4} : Trace = 4 when θ=0	Trace[**Λ**] = {4..∞} : Trace = 4 when β=0

It seems extraordinary how closely tied the cases of circular and hyperbolic motion are, until one understands that they are both just subsets of the full $\Lambda^\mu{}_\nu$ tensor formalism… Then the analogy of Bell's Spaceship to Rotating Equal Dials seems obvious...

Poincaré Transformation Equation = Inhomogeneous Lorentz = Lorentz + SpaceTime Translation:
$R^{\mu'} = \Lambda^\mu{}_\nu R^\nu + S^\mu$

Determinant[$\Lambda^{\mu'}{}_\nu$] = +1 for Proper, Continuous transforms, like rotations and boosts, but can be -1 for Improper, Discrete.
Trace[$\Lambda^{\mu'}{}_\nu$] identifies the transform type. Trace={0..4} are **R**otations, {4} is **I**dentity, {4..∞} are **B**oosts.

The combination of the factors gives : 4D Poincaré Invariance
$S^{\mu'}$: Conservation of 4-LinearMomentum **P** : SpaceTime Translation Invariance : Homogeneity in SpaceTime
$\Lambda^{\mu'}{}_\nu$: Conservation of 4-AngularMomentum ***M*** : Lorentz Invariance : Isotropy in SpaceTime

which can be separated into temporal and spatial parts.
$S^{0'}$: Conservation of Energy E : time translation invariance : homogeneity in time measurements
$S^{j'}$: Conservation of 3-linear-momentum **p** : space translation invariance : homogeneity in space measurements
$\Lambda^{\mu'}{}_\nu$ →**R**ot : Conservation of 3-angular-momentum **l**: space-space rotation invariance : isotropy in space-space measurements
$\Lambda^{\mu'}{}_\nu$ →**B**oost : Conservation of 3-mass-moment **n** : space-time boost invariance : isotropy in space-time measurements

Now, just to be completely thorough, there IS actually a way to break the string in both rotation and boost frames, including for the Invariant Interval-preserving cases, LOL.

It is not done by or because of the transformations themselves, (i.e. not by the ships pulling the strings apart), but the string-breaking can be the result of EXTREME acceleration levels. This is similar to gravitational tidal effects. In this case, it is caused by different acceleration levels along the string. Different parts of the string can and will feel different levels of force, depending on distance from Centers: (COR) or (RP). If there is an acceleration-caused force, not from ships moving apart, but purely by acceleration magnitude, that overcomes the tensile strength of the string, it will break. Thus ships could still be maintaining distance, but the string break.

For Rotations, points further from the COR, with large R, must accelerate $|\mathbf{a}| = R\Omega^2$ harder to keep the objects on a circular trajectory. A massive string, far from the COR, can possibly experience an acceleration extreme enough to break it via stress differences.
3-vectors: $|\mathbf{a}| = |\mathbf{u}|^2/|\mathbf{r}| = |a| = (R\Omega)^2/R = R\Omega^2$, with Ω=angular velocity

For Boosts, points closer to the RP, with small D, must accelerate $|\mathbf{A}| = \alpha = c^2/D$ harder to keep the objects on a hyperbolic trajectory. A massive string, close to the RP, can possibly experience an acceleration extreme enough to break it via stress differences.
4-Vectors: $|\mathbf{A}| = |\mathbf{U}|^2/|\mathbf{R}| = \alpha = c^2/D$, with c=linear velocity

Note, this is not the ships pulling the string apart. It is the stress in the string itself, caused by force differences at different points along the string, that overcomes the tensile strength of the string. Think of a heavy rope in gravitational field. Start with the rope on the ground. Start picking up the rope by one end. As more and more of the rope is lifted, the force needed to do this becomes greater and greater. Eventually, the stress at the pick up point is greater than the tensile strength of the rope can handle. Now, consider just one ship pulling a rope. Same thing, the rope could break just from an EXTREME acceleration factor.

Conclusion:

What I have done is a bit different than other people working on the problem of Bell's Spaceship Paradox. Instead of working just on the specifics of the particular Bell's Spaceship setup, I have instead generalized. I showed that there exist Invariant Intervals for event pairs (4-Displacements) that every passive-transform frame preserves for all types of Poincaré Transformations, including Boosts and Rotations. For Rotations and space-like separations in Boosts, these are proper-lengths. We know that for each passive-transformation, there is an equivalent active-transformation. So, I then generated certain types of active-transform trajectories in the initial reference frame with many possible parameters. I then calculated what the parameter values must be to make these active-transforms maintain the same space-like Invariant Interval along the entire trajectory. Then, the same invariant must hold true in any other reference frame, by the passive-transform reasoning. These would all be trajectories that ***don't break the string***, due to preservation of the proper-length. I also wrote an online program in HTML/Javascript to simulate these motions. The program numerically confirms the mathematical results by calculating Event Intervals for Rest Frames and Transformed Frames, using Rotations, Boosts, and SpaceTime Translations.

In the case of **R**otations, which are just a subset of the Poincaré Transformation:
1) Having a common Center of Rotation is the trivial solution. ex. Objects in any configuration on a circular turn-table.
2) However, there is a special case solution for different Centers of Rotation: {matching turn angles θ=ωt, matching phases (start angles), matching radii R}. This is the Spacetime Translation part of Poincaré Invariance. This would be like a string tied between the tips of two identical and synchronized dials. Other parameter combinations loosen or break the string.

In the case of **B**oosts, which are just a subset of the Poincaré Transformation:
1) Having a common Rindler Point is the trivial solution. ex. Objects in any configuration on a "hyperbolic" turn-table.
2) However, there is a special case solution for different Rindler Points: {matching acceleration rates α, matching phases (start times), matching Rindler Distances D}. This is the Spacetime Translation part of Poincaré Invariance. **This is the setup of Bell's Spaceship Paradox: Therefore, the string doesn't break.** Other parameter combinations loosen or break the string.

I wrote an online simulator that allows one to try all of these types of transformations.
http://scirealm.org/Physics-MinkowskiDiagram.html
The simulation supports these conclusions, for **R**otations and **B**oosts and SpaceTime Translations.

Also, to prove my math credentials, you can check out my online, RPN Scientific Calculator, written in Javascript. It employs complex math, has hundreds of functions (including complex Gaussian integer factorization, complex roots of unity, combinatorics, the 12-fold way, continued fraction approximations, etc.)
http://scirealm.org/SciCalculator.html

See more of my physics and math pages here:
http://scirealm.org/SiteMap.html

Going further, we can find the lines of simultaneity:

$(\Delta x, \Delta y) = (\Delta x_{COR}+r_2 \cos[\theta+p_2]-r_1 \cos[\theta+p_1], \Delta y_{COR}+r_2 \sin[\theta+p_2]-r_1 \sin[\theta+p_1])$ generally

For case 2: { already using $r_2=r_1$ and $p_2=p_1$ }
$(\Delta x, \Delta y)$
$= (\Delta x_{COR}+r_1 \cos[\theta+p_1]-r_1 \cos[\theta+p_1], \Delta y_{COR}+r_1 \sin[\theta+p_1]-r_1 \sin[\theta+p_1])$
$= (\Delta x_{COR}, \Delta y_{COR})$
$\Delta y = 0$ if $\Delta y_{COR} = 0$

For case 1: { already using $(\Delta x_{COR}, \Delta y_{COR}) =(0,0)$ }
$(\Delta x, \Delta y)$
$= (r_2 \cos[\theta+p_2]-r_1 \cos[\theta+p_1], r_2 \sin[\theta+p_2]-r_1 \sin[\theta+p_1])$
$(\Delta x, \Delta y)' = (c\Delta x + -s\Delta y, s\Delta x + c\Delta y)$
$\Delta y' = s\Delta x + c\Delta y$
$=s(r_2 \cos[\theta+p_2]-r_1 \cos[\theta+p_1])+c(r_2 \sin[\theta+p_2]-r_1 \sin[\theta+p_1])$
Let phases match $p_2=p_1$
$=s(r_2 \cos[\theta+p_1]-r_1 \cos[\theta+p_1])+c(r_2 \sin[\theta+p_1]-r_1 \sin[\theta+p_1])$
$=s(r_2 \cos[\theta']-r_1 \cos[\theta'])+c(r_2 \sin[\theta']-r_1 \sin[\theta'])$
$=s(r_2-r_1)\cos[\theta']+c(r_2-r_1)\sin[\theta']$
$=(r_2-r_1)\{s \cos[\theta']+c \sin[\theta']\}$
Let c and s have angles $[-\theta']$, an inverse rotation transformation (to see what the matching rotated frame sees)
$=(r_2-r_1)\{\sin[-\theta']\cos[\theta']+\cos[-\theta']\sin[\theta']\}$
$=(r_2-r_1)\{-\sin[\theta']\cos[\theta']+\cos[\theta']\sin[\theta']\}$
$=0$

What this means:
In the common COR frame, if phases are equal, then the rotated frame of reference sees $\Delta y'=0$, all length along $\Delta x'$
In the parallel COR frame, if phases are equal, then the start frame of reference sees $\Delta y=0$, all length along Δx
The points rise and fall along y together over all angles.

==
$(c\Delta t, \Delta x) = (c\Delta t_{RP}+d_2 \sinh[\varphi_2]-d_1 \sinh[\varphi_1], \Delta x_{RP}+d_2 \cosh[\varphi_2]-d_1 \cosh[\varphi_1])$ generally

For case 2: { already using $d_2=d_1$ and $p_2=p_1$ }
$(c\Delta t, \Delta x)$
$= (c\Delta t_{RP}+d_1 \sinh[\varphi+p_1]-d_1 \sinh[\varphi+p_1], \Delta x_{RP}+d_1 \cosh[\varphi+p_1]-d_1 \cosh[\varphi+p_1])$
$= (c\Delta t_{RP}, \Delta x_{RP})$
$\Delta t = 0$ if $\Delta t_{RP} = 0$: Rindler Points align temporally, but could vary spatially

For case 1: { already using $(c\Delta t_{RP}, \Delta x_{RP}) =(0, 0)$ }(again letting speed-of-light c=1 temporarily)
$(\Delta t, \Delta x)$
$= (d_2 \sinh[\varphi_2]-d_1 \sinh[\varphi_1], d_2 \cosh[\varphi_2]-d_1 \cosh[\varphi_1])$
$(\Delta t, \Delta x)' = (c\Delta t+s\Delta x, s\Delta t+c\Delta x)$
$\Delta t' = (c\Delta t+s\Delta x)$
$=c(d_2 \sinh[\varphi_2]-d_1 \sinh[\varphi_1])+s(d_2 \cosh[\varphi_2]-d_1 \cosh[\varphi_1])$
$=c(d_2 \sinh[\varphi+p_2]-d_1 \sinh[\varphi+p_1])+s(d_2 \cosh[\varphi+p_2]-d_1 \cosh[\varphi+p_1])$
Let phases match $p_2=p_1$
$=c(d_2 \sinh[\varphi']-d_1 \sinh[\varphi'])+s(d_2 \cosh[\varphi']-d_1 \cosh[\varphi'])$
$=\sinh[\varphi']c(d_2-d_1)+\cosh[\varphi']s(d_2-d_1)$
$=(d_2-d_1)\{\sinh[\varphi']c+\cosh[\varphi']s\}$
Let c and s have hyperangles $[-\varphi']$, an inverse transformation (to see what the matching boosted frame sees)
$=(d_2-d_1)\{\sinh[\varphi']\cosh[-\varphi']+\cosh[\varphi']\sinh[-\varphi']\}$
$=(d_2-d_1)\{\sinh[\varphi']\cosh[\varphi']-\cosh[\varphi']\sinh[\varphi']\}$
$=0$

What this means:
In the common RP frame, if phases are equal, then the boosted frame of reference sees $\Delta t'=0$=simultaneity, $\Delta x'$=proper length
In the parallel RP frame=start, if phases are equal, then the start frame of reference sees $\Delta t=0$=simultaneity, Δx=proper length
Over the entire trajectory, proper length is maintained for the type of path.

Problems with other papers on Bell's Spaceship Paradox:

Edmond M. Dewan, Michael J. Beran - NOTE ON STRESS EFFECTS DUE TO RELATIVISTIC CONTRACTION
20 March 1959

"Two unconnected objects which move in a prescribed manner with respect to an inertial frame need not satisfy constraints which are defined with respect to an instantaneous rest frame."

Dewan & Beran make the misleading argument that unconnected objects and connected objects somehow obey different physics. The real actual physics is about the endpoints of the ships, and how those are measured in different frames. Those endpoints are spacetime events, specified by Lorentz Invariant 4-Positions. The difference in those position 4-Vectors defines a 4-Displacement, which is also Poincaré Invariant. It is a specious argument to say anything about whether the events are connected or unconnected. The 4-Displacement in this case is about space-like separated events, the endpoints of the ships, which have an invariant proper-length between them. The relativistic length may appear different in different frames, but the Invariant Interval gives the invariant proper-length. This is a typical property of all SR 4-Vectors. The components will vary in different frames, but the magnitude of the 4-Vector is an invariant, which all frames will agree on. In this regard, component values of tensors are "perceptual" only.

If the endpoints move in such a way as to maintain the proper length along a trajectory in one frame, then that proper length is maintained in all frames. As shown in this paper, there are at least two ways this can be accomplished, one of which is pure hyperbolic acceleration, and one of which is the setup of the Bell's Spaceship Paradox. Numerical computer simulations of both these types of motion confirm this.

The oddity is that for parallel accelerations, the Lorentz boost frame of one ship is NOT the same Lorentz boost frame of the other. There is a spacetime translation involved, which means that the full Poincaré boost math must be used.

Examine Standard Hyperbolic Boost Again (Case 1):
$(ct_1, x_1) = (ct_{RP1}+d_1 \sinh[\varphi+p_1], x_{RP1}+d_1 \cosh[\varphi+p_1]) = (ct_{RP1}+d_1 \sinh[c\tau_1/d_1+p_1], x_{RP1}+d_1 \cosh[c\tau_1/d_1+p_1])$
$(ct_2, x_2) = (ct_{RP1}+d_2 \sinh[\varphi+p_2], x_{RP1}+d_2 \cosh[\varphi+p_2]) = (ct_{RP1}+d_2 \sinh[c\tau_2/d_2+p_2], x_{RP1}+d_2 \cosh[c\tau_2/d_2+p_2])$

$(c\Delta t, \Delta x) = (d_2 \sinh[\varphi+p_2]-d_1 \sinh[\varphi+p_1], d_2 \cosh[\varphi+p_2]-d_1 \cosh[\varphi+p_1]) = (ct_2-ct_1, x_2-x_1)$

$\text{IntervalDist}^2 = -d_1^2-d_2^2+2d_1d_2\cosh(p_1-p_2)$

Simplify
$(ct_1, x_1) = (d_1 \sinh[\varphi], d_1 \cosh[\varphi]) = d_1(\sinh[\varphi], \cosh[\varphi])$
$(ct_2, x_2) = (d_2 \sinh[\varphi], d_2 \cosh[\varphi]) = d_2(\sinh[\varphi], \cosh[\varphi])$

$(c\Delta t, \Delta x)$
$= (d_2 \sinh[\varphi+p_1]-d_1 \sinh[\varphi+p_1], d_2 \cosh[\varphi+p_1]-d_1 \cosh[\varphi+p_1])$
$= (d_2-d_1)(\sinh[\varphi+p_1], \cosh[\varphi+p_1])$
$= (d_2-d_1)(\sinh[\varphi], \cosh[\varphi])$
At start, choose $\varphi=0$, then $(c\Delta t, \Delta x) = (d_2-d_1)(0, 1)$
$\text{IntervalDist}^2 = (d_2-d_1)^2(s^2-c^2) = -(d_2-d_1)^2$

$(c\Delta t, \Delta x)'$
$= (d_2-d_1)(c \sinh[\varphi] + s \cosh[\varphi], s \sinh[\varphi] + c \cosh[\varphi])$
$= (d_2-d_1)(\sinh[\varphi+\alpha], \cosh[\varphi+\alpha])$
At boost, choose $\alpha=-\varphi$, then $(c\Delta t, \Delta x)' = (d_2-d_1)(0, 1)$
$\text{IntervalDist}^2 = (d_2-d_1)^2(s^2-c^2) = -(d_2-d_1)^2$

Use Independent Centers-of-Rotation (COR_1 & COR_2).
Choose modified Radii, by setting $r_2=a*r_1$
$=r_1^2+r_2^2-2r_1r_2\cos(p_1-p_2)+2r_1\{-\Delta x_{COR}\cos(\theta+p_1)-\Delta y_{COR}\sin(\theta+p_1)\}+2r_2\{\Delta x_{COR}\cos(\theta+p_2)+\Delta y_{COR}\sin(\theta+p_2)\}+\Delta x_{COR}^2+\Delta y_{COR}^2$
$=r_1^2+(ar_1^2)-2a(r_1)^2\cos(p_1-p_2)+2r_1\{-\Delta x_{COR}\cos(\theta+p_1)-\Delta y_{COR}\sin(\theta+p_1)\}+2ar_1\{\Delta x_{COR}\cos(\theta+p_2)+\Delta y_{COR}\sin(\theta+p_2)\}+\Delta x_{COR}^2+\Delta y_{COR}^2$
$=(1+a)r_1^2-2a(r_1)^2\cos(p_1-p_2)+2r_1\{-\Delta x_{COR}\cos(\theta+p_1)-\Delta y_{COR}\sin(\theta+p_1)+a\Delta x_{COR}\cos(\theta+p_2)+a\Delta y_{COR}\sin(\theta+p_2)\}+\Delta x_{COR}^2+\Delta y_{COR}^2$
&
Choose equal phases: by setting $p_2=p_1$
$=(1+a)r_1^2-2a(r_1)^2\cos(p_1-p_1)+2r_1\{-\Delta x_{COR}\cos(\theta+p_1)-\Delta y_{COR}\sin(\theta+p_1)+a\Delta x_{COR}\cos(\theta+p_1)+a\Delta y_{COR}\sin(\theta+p_1)\}+\Delta x_{COR}^2+\Delta y_{COR}^2$
$=(1+a)r_1^2-2a(r_1)^2\cos(0)+2r_1\{(a-1)\Delta x_{COR}\cos(\theta+p_1)+(a-1)\Delta y_{COR}\sin(\theta+p_1)\}+\Delta x_{COR}^2+\Delta y_{COR}^2$
$=(1+a)r_1^2-2a(r_1)^2+2(a-1)r_1\{\Delta x_{COR}\cos(\theta+p_1)+\Delta y_{COR}\sin(\theta+p_1)\}+\Delta x_{COR}^2+\Delta y_{COR}^2$
The only way to remove angle θ dependence is a=1, meaning the radii should be equal
==
Choose equal Radii, by setting $r_2=r_1$
$=r_1^2+r_2^2-2r_1r_2\cos(p_1-p_2)+2r_1\{-\Delta x_{COR}\cos(\theta+p_1)-\Delta y_{COR}\sin(\theta+p_1)\}+2r_2\{\Delta x_{COR}\cos(\theta+p_2)+\Delta y_{COR}\sin(\theta+p_2)\}+\Delta x_{COR}^2+\Delta y_{COR}^2$
$=2r_1^2-2r_1^2\cos(p_1-p_2)+2r_1[-\Delta x_{COR}\cos(\theta+p_1)-\Delta y_{COR}\sin(\theta+p_1)+\Delta x_{COR}\cos(\theta+p_2)+\Delta y_{COR}\sin(\theta+p_2)]+\Delta x_{COR}^2+\Delta y_{COR}^2$
&
Choose modified phases: by setting $p_1=a*p_2$
$= 2r_1^2-2r_1^2\cos(p_1-ap_1)+2r_1[-\Delta x_{COR}\cos(\theta+p_1)-\Delta y_{COR}\sin(\theta+p_1)+\Delta x_{COR}\cos(\theta+ap_1)+\Delta y_{COR}\sin(\theta+ap_1)]+\Delta x_{COR}^2+\Delta y_{COR}^2$
$= 2r_1^2(1-\cos(p_1-ap_1))+2r_1[\Delta x_{COR}(\cos(\theta+ap_1)-\cos(\theta+p_1))+\Delta y_{COR}(\sin(\theta+ap_1)-\sin(\theta+p_1)))]+\Delta x_{COR}^2+\Delta y_{COR}^2$
The only way to remove angle θ dependence is a=1, meaning the phases should be equal

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General Motion using 4-Vectors:
4-Position $\mathbf{R} = R^\mu = (ct,\mathbf{r}) = \mathbf{R} = d^0\mathbf{R}/d\tau^0$
4-Velocity $\mathbf{U} = U^\mu = \gamma(c,\mathbf{u}) = d\mathbf{R}/d\tau = d^1\mathbf{R}/d\tau^1$
4-Acceleration $\mathbf{A} = A^\mu = \gamma(c\gamma',\gamma'\mathbf{u}+\gamma\mathbf{a}) = d\mathbf{U}/d\tau = d^2\mathbf{R}/d\tau^2$

All Lorentz Scalar Products are Invariants
$(\mathbf{R}\cdot\mathbf{R}) = (ct)^2 - \mathbf{r}\cdot\mathbf{r} = (ct_o)^2 = (c\tau)^2 = -(\mathbf{r_o}\cdot\mathbf{r_o})$:either (±),variable
$(\mathbf{U}\cdot\mathbf{U}) = (c)^2$:temporal(+),fundamental constant
$(\mathbf{A}\cdot\mathbf{A}) = -(a_o)^2 = -(\alpha)^2 = (i\alpha)^2$:spatial(–),variable

Linear Inertial Motion using 4-Vectors:
4-Position $\mathbf{R} = R^\mu = (ct+ct_i,\mathbf{r}=\mathbf{u}_i t+\mathbf{r}_i) = \mathbf{R}$
4-Velocity $\mathbf{U} = U^\mu = \gamma_i(c,\mathbf{u}_i) = d\mathbf{R}/d\tau = \gamma d\mathbf{R}/dt$
4-Acceleration $\mathbf{A} = A^\mu = (0,\mathbf{0}) = d\mathbf{U}/d\tau = \gamma d\mathbf{U}/dt$

Calculations for Active-Transform Boost Path/Worldline:
Pick a totally arbitrary frame-origin point (ct_o , x_o) and place the ships arbitrarily.
Then, we want the two points to move in such a way as to maintain interval regardless of frame-time.

Calculate for event points using Independent Initial StartPoints, the constants $(ct_{1i} , \mathbf{r}_{1i})$ & $(ct_{2i} , \mathbf{r}_{2i})$:
The coordinates of the two points in a lab reference frame are:
$(ct_1 , \mathbf{r}_1) = (ct_o+ct+ct_{i1} , x_o+\mathbf{u}_{i1}t+\mathbf{r}_{i1})$
$(ct_2 , \mathbf{r}_2) = (ct_o+ct+ct_{i2} , x_o+\mathbf{u}_{i2}t+\mathbf{r}_{i2})$

The displacement between the points is:
$(c\Delta t , \Delta\mathbf{r})$
$= (ct_2-ct_1 , \mathbf{r}_2-\mathbf{r}_1)$
$= (ct_o+ct+ct_{i2}-ct_o-ct-ct_{i1} , x_o+\mathbf{u}_{i2}t+\mathbf{r}_{i2}-x_o-\mathbf{u}_{i1}t-\mathbf{r}_{i1})$
$= (ct_{i2}-ct_{i1} , \mathbf{u}_{i2}t+\mathbf{r}_{i2}-\mathbf{u}_{i1}t-\mathbf{r}_{i1})$ ie. the frame-origin point (ct_o , x_o) doesn't matter (SpaceTime Translation Invariance)
$= (c\{t_{i2}-t_{i1}\}, \{\mathbf{u}_{i2}-\mathbf{u}_{i1}\}t+\{\mathbf{r}_{i2}-\mathbf{r}_{i1}\})$
$= (c\Delta t_i, \Delta\mathbf{u}_i t+\Delta\mathbf{r}_i)$

IntervalDist^2 {4D Invariant}
$= (c^2\Delta t^2-\Delta\mathbf{r}^2) = (c^2\Delta t'^2-\Delta\mathbf{r}'^2)$
$=\{c\Delta t_i\}^2-\{\{\Delta\mathbf{u}_i\}t+\{\Delta\mathbf{r}_i\}\}^2$
$=\{c\Delta t_i\}^2-\{\{\Delta\mathbf{u}_i t\}^2+2\{\Delta\mathbf{u}_i t\}\{\Delta\mathbf{r}_i\}+\{\Delta\mathbf{r}_i\}^2\}$
$=\{c\Delta t_i\}^2-\{\Delta\mathbf{r}_i\}^2-\{\Delta\mathbf{u}_i t\}^2-2\{\Delta\mathbf{u}_i t\Delta\mathbf{r}_i\}$

which is not generally independent of path frame-time (t).

Now, how can we make this independent of the path parameter, i.e. frame-time (t), so a string between won't break?

The 1st way:
Choose a Common Velocity, by setting $\Delta\mathbf{u}_i=\mathbf{0}$. The velocity vectors are in the same direction and equal in magnitude.
Giving $(c\Delta t , \Delta\mathbf{r}) = (c\Delta t_i , \Delta\mathbf{r}_i)$, a constant 4-Displacement
Then: IntervalDist^2 = $\{c\Delta t_i\}^2-\{\Delta\mathbf{r}_i\}^2$, which is also just a constant and independent of the path frame-time (t)
This is pure inertial motion of two event points.
This could be the motion of two event points which are displaced in both space AND time.
They will maintain a constant Invariant Interval along the entire trajectory of (t).

For a physical object, the endpoints are space-like separated, and the 4-Displacement is spatial = InvariantInterval(-).
Proper-length $\Delta\mathbf{r_o}$ will be measured with IntervalDist^2 = $\{c\Delta t_i\}^2-\{\Delta\mathbf{r}_i\}^2$ being purely spatial.
In this case, it is the temporal part of the 4D invariant which is set to zero, ie. it is simultaneous, $\Delta t_i=0$ and $\Delta\mathbf{r}_i \rightarrow \Delta\mathbf{r_o}$
giving IntervalDist^2 = $-\{\Delta\mathbf{r_o}\}^2$ which is a technically a simultaneity-length, but could be called a "rest-frame" length.

The SR idea of a rest-frame is used for 4-Vectors which are temporal = InvariantInterval(+), like 4-Velocity **U** & 4-Momentum **P**.
In this case, it is the spatial part of the 4D invariant which is set to zero = it is not moving spatially = it is at rest = rest frame.

4-Velocity $\mathbf{U} = U^\mu = \gamma(c,\mathbf{u})$
$(\mathbf{U}\cdot\mathbf{U}) = \gamma(c,\mathbf{u})\cdot\gamma(c,\mathbf{u}) = \gamma^2(c^2-\mathbf{u}\cdot\mathbf{u}) = c^2\gamma^2(1-\mathbf{u}\cdot\mathbf{u}/c^2) = (c)^2 = (c^2-\mathbf{0}\cdot\mathbf{0}) = (c)^2$, which is the same as rest frame $(c,\mathbf{0})_{rest}\cdot(c,\mathbf{0})_{rest} = (c)^2$

4-Momentum $\mathbf{P} = P^\mu = (E/c,\mathbf{p}) = (E_o/c^2)\mathbf{U} = (m_o)\mathbf{U}$
The tensor invariant way: $(\mathbf{P}\cdot\mathbf{P}) = (E_o/c^2)\mathbf{U}\cdot(E_o/c^2)\mathbf{U} = (E_o/c^2)^2\mathbf{U}\cdot\mathbf{U} = (E_o/c^2)^2(c)^2 = (E_o/c)^2 = (m_oc)^2$, a rest energy:mass
The scalar product rest frame way: $(\mathbf{P}\cdot\mathbf{P}) = (E/c,\mathbf{p})\cdot(E/c,\mathbf{p}) = (E/c)^2-\mathbf{p}\cdot\mathbf{p} = (E_o/c)^2-\mathbf{0}\cdot\mathbf{0} = (E_o/c)^2 = (m_oc)^2$, a rest energy:mass
The rest frame way: $(\mathbf{P}\cdot\mathbf{P}) = (E_o/c,\mathbf{0})_{rest}\cdot(E_o/c,\mathbf{0})_{rest} = (E_o/c)^2 = (m_oc)^2$, a rest energy:mass